NEW CUBIC FOURFOLDS WITH ODD-DEGREE UNIRATIONAL PARAMETRIZATIONS

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Abstract. We prove that the moduli space of cubic fourfolds C contains a divisor C_{42} whose general member has a unirational parametrization of degree 13. This result follows from a thorough study of the Hilbert scheme of rational scrolls and an explicit construction of examples. We also show that C_{42} is uniruled.

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Introduction

Let X be a smooth projective variety of dimension n over $\mathbb{C}$. We say that X has a *degree ϱ unirational parametrization* if there is a dominant rational map $\rho : \mathbb{P}^n \dashrightarrow X$ with $\deg \rho = \varrho$. Such a parametrization implies that the smallest positive integer N which allows the rational equivalence

$$N\Delta_X \equiv N\{x \times X\} + Z \quad \text{in} \quad \mathrm{CH}^n(X \times X) \tag{0.1}$$

would divide ϱ, where $x \in X$ and Z is a cycle supported on $X \times Y$ for some divisor $Y \subset X$. The relation (0.1) for arbitrary integer N is called a *decomposition of the diagonal* of X, and it is called an *integral* decomposition of the diagonal if $N = 1$. ([BS83]. See also [Voi14, Chap. 3].)

This paper studies the unirationality of *cubic fourfolds*, i.e. smooth cubic hypersurfaces in $\mathbb{P}^5$ over $\mathbb{C}$. Let $\mathrm{Hdg}^4(X, \mathbb{Z}) := H^4(X, \mathbb{Z}) \cap H^2(\Omega_X^2)$ be the group of integral Hodge classes of degree 4 for a cubic fourfold X. In the coarse moduli space of cubic fourfolds C, the Noether-Lefschetz locus

$$\left\{X \in C \,:\, \mathrm{rk}\left(\mathrm{Hdg}^4(X, \mathbb{Z})\right) \geq 2\right\}$$

is a countably infinite union of irreducible divisors C_d indexed by $d \geq 8$ and $d \equiv 0, 2 \pmod 6$. Here C_d consists of the *special cubic fourfolds* which admit a rank-2 saturated sublattice of discriminant d in $\mathrm{Hdg}^4(X, \mathbb{Z})$ [Has00]. Because the integral Hodge conjecture is valid for cubic fourfolds [Voi13, Th. 1.4], $X \in C$ is special if and only if there is an algebraic surface $S \subset X$ not homologous to a complete intersection.

Voisin [Voi17, Th. 5.6] proves that a special cubic fourfold of discriminant $d \equiv 2 \pmod 4$ admits an integral decomposition of the diagonal. Because every cubic fourfold has a unirational parametrization of degree 2 [Har95, Example 18.19], it is natural to ask whether they have odd degree unirational parametrizations.

For a general $X \in C_d$ with $d = 14$, 18, 26, 30, and 38, the examples constructed by Nuer [Nue16] combined with an algorithm by Hassett [Has16, Prop. 38] support the expectation. In this paper, we improve the list by solving the case $d = 42$.

Theorem 0.1. *A generic $X \in C_{42}$ has a degree 13 unirational parametrization.*

Recall that a variety Y is *uniruled* if there is a variety Z and a dominant rational map $Z \times \mathbb{P}^1 \dashrightarrow Y$ which doesn't factor through the projection to Z. As a byproduct of the proof of Theorem 0.1, we also prove that

Theorem 0.2. *C_{42} is uniruled.*

Strategy of Proof. When $d = 2(n^2 + n + 1)$ with $n \geq 2$ and $X \in C_d$ is general, the Fano variety of lines $F_1(X)$ is isomorphic to the Hilbert scheme of two points $\Sigma^{[2]}$, where Σ is a K3 surface polarized by a primitive ample line bundle of degree d. [Has00, Th. 6.1.4]

The isomorphism $F_1(X) \cong \Sigma^{[2]}$ implies that X contains a family of two-dimensional rational scrolls parametrized by Σ. Indeed, the divisor $\Delta \subset \Sigma^{[2]}$ parametrizing the non-reduced subschemes can be naturally identified as the projectivization of the tangent bundle of Σ. Each fiber of this $\mathbb{P}^1$-bundle induces a smooth rational curve in $F_1(X)$ through the isomorphism, hence corresponds to a rational scroll in X.

Let $S \subset X$ be one of such scrolls. Since S is rational, its symmetric square $W = \mathrm{Sym}^2 S$ is also rational. A generic element $s_1 + s_2 \in W$ spans a line $l(s_1, s_2)$ not contained in X, so there is a rational map

$$\begin{array}{rccc} \rho : & W & \dashrightarrow & X \\ & s_1 + s_2 & \mapsto & x \end{array}$$

where $l(s_1, s_2) \cap X = \{s_1, s_2, x\}$. By [Has16, Prop. 38], this map becomes a unirational parametrization if S has isolated singularities. Moreover, its degree is odd as long as $4 \nmid d$.

Discriminant $d = 42$ corresponds to the case $n = 4$ above. Note that $4 \nmid d = 42$. Thus a generic $X \in C_{42}$ admits an odd degree unirational parametrization once we prove that

Theorem 0.3. *A generic $X \in C_{42}$ contains a degree-9 rational scroll S which has 8 double points and is smooth otherwise.*

Here a *double point* means a *non-normal ordinary double point*. It's a point where the surface has two branches that meet transversally.

The idea in proving Theorem 0.3 is as follows:

Degree-9 scrolls in $\mathbb{P}^5$ form a component $\mathcal{H}_9$ in the associated Hilbert scheme. Let $\mathcal{H}_9^8 \subset \mathcal{H}_9$ parametrize scrolls with 8 isolated singularities. By definition (See Section 4) an element $\overline{S} \in \mathcal{H}_9^8$ is non-reduced. We use S to denote its underlying variety.

Let $U_{42} \subset |\mathcal{O}_{\mathbb{P}^5}(3)|$ be the locus of special cubic fourfolds with discriminant 42. Consider the incidence variety

$$\mathcal{Z} = \left\{ (\overline{S}, X) \in \mathcal{H}_9^8 \times U_{42} : S \subset X \right\}.$$

Then there is a diagram

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\ \mathcal{H}_9^8 & & U_{42} \arrow[r] & C_{42} \end{tikzcd}$$

Theorem 0.3 is proved by showing that p_2 is dominant. Two main ingredients in the proof are:

- Constructing an explicit example.
- Estimating the dimension of the Hilbert scheme parametrizing singular scrolls.

Section 1 provides an introduction of rational scrolls and the basic properties required in the proof. We construct an example in Section 2 and then prove the main results in Section 3. The general description about the

Hilbert schemes $\mathcal{H}_9^8 \subset \mathcal{H}_9$ and the estimate of the dimensions are left to Section 4.

Throughout the paper we will frequently deal with the rational map

$$\Lambda_Q : \mathbb{P}^{D+1} \dashrightarrow \mathbb{P}^N$$

defined as the projection from some $(D-N)$-plane Q. Here D and N are positive integers such that $D+1 \geq N \geq 3$. We will assume $D \geq N \geq 5$ when we are studying singular scrolls.

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1. Preliminary: rational scrolls

We provide a brief review of rational scrolls and introduce necessary terminologies and lemmas in this section.

1.1. Hirzebruch surfaces.

Let m be a nonnegative integer, and let $\mathscr{E}$ be a rank two locally free sheaf on $\mathbb{P}^1$ isomorphic to $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m)$. The Hirzebruch surface $\mathbb{F}_m$ is defined to be the associated projective space bundle $\mathbb{P}(\mathscr{E})$.

Let f be the divisor class of a fiber, and let g be the divisor class of a section, i.e. the divisor class associated with Serre's twisting sheaf $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)$. The Picard group of $\mathbb{F}_m$ is freely generated by f and g, and the intersection pairing is given by

	f	g
f	0	1
g	1	m.

The canonical divisor is $K_{\mathbb{F}_m} = -2g + (m-2)f$.

Let a and b be two integers, and let $h = ag + bf$ be a divisor on $\mathbb{F}_m$. The ampleness and the very ampleness for h are equivalent on $\mathbb{F}_m$, and it happens if and only if $a > 0$ and $b > 0$. [Har77, Chapter V, §2.18]

Lemma 1.1. *Suppose the divisor $ag + bf$ is ample. We have*

$$\begin{aligned} h^0(\mathbb{F}_m, ag+bf) &= (a+1)\left(\tfrac{1}{2}am + b + 1\right), \\ h^i(\mathbb{F}_m, ag+bf) &= 0 \quad \textit{for all } i > 0. \end{aligned}$$

These formulas appear in several places in the literature with slightly different details depending on the contexts, for example [Laf02, Prop. 2.3], [BBF04, p.543], and [Cos06, Lemma 2.6]. It can be proved by induction on the integers a and b or by applying the projection formula to the bundle map $\pi : \mathbb{F}_m \to \mathbb{P}^1$.

1.2. **Deformations of Hirzebruch surfaces.** $\mathbb{F}_m$ admits a deformation to $\mathbb{F}_{m-2k}$ for all $m > 2k \geq 0$. More precisely, there exitsts a holomorphic family $\tau : \mathcal{F} \to \mathbb{C}$ such that $\mathcal{F}_0 \cong \mathbb{F}_m$ and $\mathcal{F}_t \cong \mathbb{F}_{m-2k}$ for $t \neq 0$. The family can be written down explicitly by the equation

$$\mathcal{F} = \left\{ {x_0}^m y_1 - {x_1}^m y_2 + t {x_0}^{m-k} {x_1}^k y_0 = 0 \right\} \subset \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{C}, \tag{1.1}$$

where $([x_0, x_1], [y_0, y_1, y_2], t)$ is the coordinate of $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{C}$. [BPVdV84, p.205]

Generally, $\mathbb{F}_m$ admits an analytic versal deformation with a base manifold of dimension $h^1(\mathbb{F}_m, T_{\mathbb{F}_m})$ by the following lemma.

Lemma 1.2. [Sei92, Lemma 1. and Theorem 4.] *There is a natural isomorphism $H^1(\mathbb{F}_m, T_{\mathbb{F}_m}) \cong H^1\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-m)\right)$. We also have $H^2(\mathbb{F}_m, T_{\mathbb{F}_m}) = 0$.*

Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m)$ be the underlying locally free sheaf of $\mathbb{F}_m$. It is straightforward to compute that ${\rm Ext}^1{}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{E}) \cong H^1\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-m)\right)$, so there is a natural isomorphism

$$H^1(\mathbb{F}_m, T_{\mathbb{F}_m}) \cong {\rm Ext}^1_{\mathbb{P}^1}(\mathcal{E}, \mathcal{E}), \tag{1.2}$$

by Lemma 1.2. The elements of the group ${\rm Ext}^1{}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{E})$ are in one-to-one correspondence with the deformations of $\mathcal{E}$ over the dual numbers $D_t \cong \frac{\mathbb{C}[t]}{(t^2)}$. [Har10, Th. 2.7] Thus (1.2) says that the infinitesimal deformation of $\mathbb{F}_m$ can be identified with the infinitesimal deformation of its underlying locally free sheaf.

Every element in ${\rm Ext}^1{}_{\mathbb{P}^1}(\mathcal{E}, \mathcal{E}) \cong {\rm Ext}^1{}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(m), \mathcal{O}_{\mathbb{P}^1})$ is represented by a short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(m-k) \to \mathcal{O}_{\mathbb{P}^1}(m) \to 0$$

for some k satisfying $m > 2k \geq 0$. By tracking the construction of the correspondence in [Har10, Th. 2.7], the above sequence corresponds to a coherent sheaf $\mathscr{E}$ on $\mathbb{P}^1 \times D_t$, flat over D_t, such that $\mathscr{E}_0 \cong \mathcal{E}$ and $\mathscr{E}_t \cong \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(m-k)$ for $t \neq 0$. So it induces a flat family $\mathcal{F}$ of Hirzebruch surfaces over D_t such that $\mathcal{F}_0 \cong \mathbb{F}_m$ and $\mathcal{F}_t \cong \mathbb{F}_{m-2k}$ for $t \neq 0$.

1.3. **Rational normal scrolls.**

Let u and v be positive integers with $u \leq v$ and let $N = u + v + 1$. Let P_1 and P_2 be complementary linear subspaces of dimensions u and v in $\mathbb{P}^N$. Choose rational normal curves $C_1 \subset P_1$, $C_2 \subset P_2$, and an isomorphism $\varphi : C_1 \to C_2$. Then the union of the lines $\bigcup_{p \in C_1} \overline{p\,\varphi(p)}$ forms a smooth surface $S_{u,v}$ called a *rational normal scroll of type* (u, v). The line $\overline{p\,\varphi(p)}$ is called a *ruling*. When $u < v$, we call the curve $C_1 \subset S_{u,v}$ the *directrix* of $S_{u,v}$.

A rational normal scroll of type (u, v) is uniquely determined up to projective isomorphism. In particular, each $S_{u,v}$ is projectively equivalent to the one given by the parametric equation

$$\begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & \mathbb{P}^N \\ (s,t) & \longmapsto & (1, s, ..., s^u, t, st, ..., s^v t). \end{array} \tag{1.3}$$

One can check by this expression that a hyperplane section of $S_{u,v}$ which doesn't contain a ruling is a rational normal curve of degree $u + v$. It easily follows that $S_{u,v}$ has degree $D = u + v$.

The rulings of $S_{u,v}$ form a rational curve in $\mathbb{G}(1, N)$ the Grassmannian of lines in $\mathbb{P}^N$. By using (1.3), we can parametrize this curve as

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{G}(1, N) \\ s & \longmapsto & \begin{pmatrix} 1 & s & ... & s^u & 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 & 1 & s & ... & s^v \end{pmatrix}, \end{array} \tag{1.4}$$

where the matrix on the right represents the line spanned by the row vectors.

The embedding $S_{u,v} \subset \mathbb{P}^N$ can be seen as the Hirzebruch surface $\mathbb{F}_{v-u}$ embedded in $\mathbb{P}^N$ through the complete linear system $|g + uf|$. Conversely, every nondegenerate, irreducible and smooth surface of degree D in $\mathbb{P}^{D+1}$ isomorphic to $\mathbb{F}_{v-u}$ must be $S_{u,v}$.[GH94, p.522-525]

It's not hard to compute that $H^1(T_{\mathbb{P}^N}|_{\mathbb{F}_{u-v}}) = 0$ under the above embedding. Combining this with the rigidity result, it implies that every abstract deformation of $\mathbb{F}_{v-u}$ can be lifted to an embedded deformation as a family of rational normal scrolls in $\mathbb{P}^N$. [Har10, Remark 20.2.1] We conclude this as the following lemma:

Lemma 1.3. *For $m > 2k \geq 0$, let $\mathcal{F}$ be an abstract deformation of Hirzebruch surfaces such that $\mathcal{F}_0 \cong \mathbb{F}_m$ and $\mathcal{F}_t \cong \mathbb{F}_{m-2k}$ for $t \neq 0$. Then $\mathcal{F}$ can be realized as an embedded deformation $\mathcal{S}$ in $\mathbb{P}^{D+1}$ with $\mathcal{S}_0 \cong S_{u,v}$ and $\mathcal{S}_t \cong S_{u+k,v-k}$ for $t \neq 0$, where $D = u + v$, and $u \leq v$ are any positive integers satisfying $v - u = m$.*

1.4. **Rational scrolls.**

Definition 1.4. *We call a surface $S \subset \mathbb{P}^N$ a rational scroll (or a scroll) of type $(u, m+u)$ if it is the image of a Hirzebruch surface $\mathbb{F}_m$ through a birational morphism defined by an N-dimensional subsystem $\mathfrak{d} \subset |g + uf|$ for some $u > 0$.*

Equivalently, $S \subset \mathbb{P}^N$ is a rational scroll of type (u, v) either if it is a rational normal scroll $S_{u,v}$, or if it is the projection image of $S_{u,v} \subset \mathbb{P}^{D+1}$ from a $(D-N)$-plane disjoint from $S_{u,v}$. Here $D = u + v$ is the degree of $S_{u,v}$ as well as the degree of S. In the latter case, we also call a line on S a ruling if its preimage is a ruling on $S_{u,v}$.

The following lemma computes the cohomology groups of the normal bundle for an arbitrary embedding of a Hirzebruch surface into a projective space.

Lemma 1.5. *Let $\iota : \mathbb{F}_m \hookrightarrow \mathbb{P}^N$ be an embedding with image S and $\iota^* \mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{O}_S(h)$, where $h = ag + bf$ with $a > 0$ and $b > 0$. Let $N_{S/\mathbb{P}^N}$ be the normal bundle of S in $\mathbb{P}^N$, then*

$$h^0(S, N_{S/\mathbb{P}^N}) = (N+1)(a+1)(\frac{am}{2} + b + 1) - 7$$

and $h^i(S, N_{S/\mathbb{P}^N}) = 0, \forall i > 0$. Especially, if S is a smooth scroll of degree D, then the formula for h^0 reduces to

$$h^0(S, N_{S/\mathbb{P}^N}) = (N+1)(D+2) - 7.$$

Proof. The short exact sequence

$$0 \to T_S \to T_{\mathbb{P}^N}|_S \to N_{S/\mathbb{P}^N} \to 0 \tag{1.5}$$

has the long exact sequence

$$\begin{array}{ccccccccc} 0 & \to & H^0(S, T_S) & \to & H^0(S, T_{\mathbb{P}^N}|_S) & \to & H^0(S, N_{S/\mathbb{P}^N}) & & \\ & \to & H^1(S, T_S) & \to & H^1(S, T_{\mathbb{P}^N}|_S) & \to & H^1(S, N_{S/\mathbb{P}^N}) & & \\ & \to & H^2(S, T_S) & \to & H^2(S, T_{\mathbb{P}^N}|_S) & \to & H^2(S, N_{S/\mathbb{P}^N}) & \to & 0. \end{array}$$

In order to calculate the dimensions in the right, we need the dimensions in the first two columns.

For the middle column, we can restrict the Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_{\mathbb{P}^N}(1)^{\oplus(N+1)} \to T_{\mathbb{P}^N} \to 0$$

to S and obtain

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(h)^{\oplus(N+1)} \to T_{\mathbb{P}^N}|_S \to 0.$$ [1]

[1] $\mathrm{Tor}_i^{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_S, \mathscr{F}) = 0$ for all $i > 0$ and locally free sheaf $\mathscr{F}$, so the Euler exact sequence keeps exact after the restriction.

Lemma 1.1 confirms that $h^i(S, \mathcal{O}_S(h)) = 0$ for $i > 0$, so we have

$$0 \to H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{O}_S(h))^{\oplus(N+1)} \to H^0(S, T_{\mathbb{P}^N}|_S) \to 0$$

from the associated long exact sequence while the other terms are all vanishing. It follows that

$$\begin{aligned} h^0(S, T_{\mathbb{P}^N}|_S) &= (N+1)h^0(S, \mathcal{O}_S(h)) - h^0(S, \mathcal{O}_S) \\ &\overset{1.1}{=} (N+1)(a+1)(\tfrac{am}{2} + b + 1) - 1. \end{aligned}$$

For the first column, one can use the Hirzebruch-Riemann-Roch formula to compute that $\chi(T_S) = 6$. We also have $h^2(S, T_S) = 0$ by Lemma 1.2. Thus $h^0(S, T_S) - h^1(S, T_S) = \chi(T_S) = 6$.

Collecting the above results, the long exact sequence for (1.5) now becomes

$$\begin{array}{ccccccccc} 0 & \to & H^0(S, T_S) & \to & H^0(S, T_{\mathbb{P}^N}|_S) & \to & H^0(S, N_{S/\mathbb{P}^N}) & & \\ & \to & H^1(S, T_S) & \to & 0 & \to & H^1(S, N_{S/\mathbb{P}^N}) & & \\ & \to & 0 & \to & 0 & \to & H^2(S, N_{S/\mathbb{P}^N}) & \to & 0. \end{array}$$

Therefore we have $h^i(S, N_{S/\mathbb{P}^N}) = 0, \forall i > 0$, and

$$\begin{aligned} h^0(S, N_{S/\mathbb{P}^N}) &= h^0(S, T_{\mathbb{P}^N}|_S) - \chi(T_S) \\ &= (N+1)(a+1)(\tfrac{am}{2} + b + 1) - 7. \end{aligned}$$

When S is a rational scroll, we have $h = g + bf$. Then the formula is obtained by inserting $a = 0$ and $D = h^2 = m + 2b$ into the equation. ◻

1.5. **Isolated singularities on rational scrolls.** The singularities on a rational scroll are all caused from projection by definition, so we assume $D \geq N$. We also assume $N \geq 5$.

Let $S \subset \mathbb{P}^N$ be a rational scroll under $S_{u,v} \subset \mathbb{P}^{D+1}$ and let $q : S_{u,v} \to S$ be the projection. A point $p \in S$ is singular if and only if one of the following situations occurs

- There are two distinct rulings $l, l' \subset S_{u,v}$ such that $p \in q(l) \cap q(l')$.
- There is a ruling $l \subset S_{u,v}$ such that $p \in q(l)$ and the map q is ramified at l.

Suppose that S has isolated singularities, i.e. the singular locus of S has dimension zero. Then each singular point is set-theoretically the intersection of two or more rulings. Let m be the number of the ruling which passes through any of the singular points. Then the number of singularities on S is counted as $\binom{m}{2}$.

Note that $S_{u,v}$ is cut out by quadrics so every secant line intersects $S_{u,v}$ in exactly two points transversally. Let $T(S_{u,v}) \subset S(S_{u,v})$ respectively be the tangent and the secant varieties of $S_{u,v}$. Then every $x \in S(S_{u,v}) - T(S_{u,v})$ belongs to one of the two conditions

(1) The point x lies on one and only one secant line.
(2) The point x lies on two secant lines. Let $Z_2 \subset S(S_{u,v})$ denote the union of such points.

Lemma 1.6. *The subset $Z_2 \neq \emptyset$ if and only if $u = 2$. In this situation, $Z_2 \cong \mathbb{P}^2$ and each $x \in Z_2$ lies on infinitely many secant lines.*

Proof. We retain the notation used in constructing $S_{u,v}$ throughout the proof.

Let $x \in Z_2$ be any point. First we claim that the intersection points of $S_{u,v}$ with the union of the two secants described in (2) lie on four distinct rulings.

The intersection points don't lie on two rulings because any two distinct rulings are linearly independent.[Har95, Exercise 8.21] If they lie on three rulings, then the projection to P_2 would be a trisecant line of C_2. But this is impossible because C_2 has degree $v \geq \lceil \frac{D}{2} \rceil \geq 2$. Hence the claim holds.

The claim admits a rational normal curve $C \subset S_{u,v}$ (either sectional or residual) of degree $\geq u$ passing through the four intersection points.[Har95, Example 8.17] This imposes a non-trivial linear relation on four distinct points on C, which forces C to be either a line or a conic. If C is a line then C coincides with the two secant lines, which is impossible. Hence C must be a conic..

It follows that $u \leq \deg C \leq 2$. If $u = 1$, then the conic C would dominate C_2 through the projection from P_1. However, this cannot happen since C_2 has degree $v = D - u \geq 4$. Hence $u = 2$. In this condition, C can only be the directrix since $u < v$. It follows that the Z_2 coincides with the 2-plane spanned by C and each point of Z_2 lies on infinitely many secants.

Conversely, $u = 2$ implies that Z_2 contains the 2-plane spanned by C_2. By the same arguement above they coincide and every $x \in Z_2$ lies on infinitely many secants. □

Assume that S is the projection of $S_{u,v}$ from a $(D - N)$-plane $Q \subset \mathbb{P}^{D+1}$.

Corollary 1.7. *The scroll S is singular along r points if and only if Q intersects $S(S_{u,v})$ in r points away from $T(S_{u,v}) \cup Z_2$.*

Proof. Assume that S has isolated singularities. Recall that the number r counts the number of the pair (l, l') of distinct rulings on $S_{u,v}$ such that $q(l)$ intersects $q(l')$ in one point. (Different pairs might intersect in the same point.) It then counts the number of the unique line joining l, l' and Q. By

Lemma 1.6, each $x \in S(S_{u,v})$ away from $T(S_{u,v}) \cup Z_2$ lies on a unique secant. Thus it is the same as the number of the intersection between Q and $S(S_{u,v})$ away from $T(S_{u,v}) \cup Z_2$. $\square$

In the end we provide a criterion for S to have isolated singularities when $u = 1$. This is going to be used in proving Proposition 2.1.

Proposition 1.8. *Assume $u = 1$. If $Q \cap T(S_{1,v}) = \emptyset$, then S has isolated singularities.*

Proof. If Q intersects $S(S_{1,v})$ in points, then the proposition follows from Corollary 1.7.

Assume $Q \cap S(S_{1,v})$ contains a curve Γ. We are going to show that Γ intersects $T(S_{1,v})$ nontrivially which then contradicts our hypothesis.

Let f be the fiber class of $S_{1,v}$. Then the linear system $|f|$ parametrizes the rulings of $S_{1,v}$. For distinct $l, l' \in |f|$, the linear span of l and l' is a 3-space $P_{l,l'} \subset S(S_{1,v})$. Consider the incidence correspondence

$$\mathbb{S} = \{(x, l + l') \in \mathbb{P}^{D+1} \times |2f| : x \in P_{l,l'}\}.$$

Observe that $\mathbb{S}$ is a $\mathbb{P}^3$-bundle over $|2f| \cong \mathbb{P}^2$ via the second projection

$$p_2 : \mathbb{S} \to |2f|.$$

On the other hand, the image of $\mathbb{S}$ under the first projection

$$p_1 : \mathbb{S} \to \mathbb{P}^{D+1}$$

is the secant variety $S(S_{1,v})$. Consider the diagonal

$$\Delta := \{2l : l \in |f|\} \subset |2f|.$$

It's easy to see that the tangent variety $T(S_{1,v}) \subset S(S_{1,v})$ is the image of $p_2^{-1}(\Delta)$ via the first projection.

If $\Gamma \not\subset P_{l,l'}$ for all $l + l'$, then the curve $p_1^{-1}(\Gamma)$ is mapped to a curve in $|2f|$ which intersects Δ nontrivially. It follows that $\Gamma \cap T(S_{u,v}) \neq \emptyset$.

Suppose $\Gamma \subset P_{l,l'}$ for some $l + l'$. The directrix C_1 is a line in $P_{l,l'}$ by hypothesis, so we have

$$T(S_{u,v}) \cap P_{l,l'} = P \cup P'$$

where P and P' are the 2-planes spanned by C_1 and l and l', respectively. So Γ and $T(S_{u,v})$ has a nontrivial intersection in $P_{l,l'}$. $\square$

2. Construction of singular scrolls in $\mathbf{P}^5$

This section provides a construction of singular scrolls in $\mathbb{P}^5$ of type $(1, v)$ with isolated singularities. The construction actually relates the existence of the singular scrolls to the solvability of a four-square equation as follows:

Proposition 2.1. *Assume $v \geq 4$. There exists a rational scroll in $\mathbb{P}^5$ of type $(1, v)$ with isolated singularities which has at least r singularities if there are four odd integers $a \geq b \geq c \geq d > 0$ satisfying*

(1) $8r + 4 = a^2 + b^2 + c^2 + d^2$,
(2) $a + b + c \leq 2v - 3$.

We use the construction to produce an explicit example which can be manipulated by a computer algebra system. With the help of a computer we prove that

Proposition 2.2. *There is a degree-9 rational scroll $S \subset \mathbb{P}^5$ which has eight isolated singularities and smooth otherwise such that*

(1) $h^0(\mathbb{P}^5, \mathcal{I}_S(3)) = 6$, *where $\mathcal{I}_S$ is the ideal sheaf of S in $\mathbb{P}^5$.*
(2) *S is contained in a smooth cubic fourfold X.*
(3) *S deforms in X to the first order as a two dimensional family.*
(4) *S is also contained in a singular cubic fourfold Y.*

We introduce the construction first and prove Proposition 2.2 in the end. Recall that, with a fixed rational normal scroll $S_{1,v} \subset \mathbb{P}^{D+1}$, every degree D scroll $S \subset \mathbb{P}^5$ of type $(1, v)$ is the projection of $S_{1,v}$ from $P^{\perp}$ for some $P \in \mathbb{G}(5, D + 1)$.

2.1. **Plane k-chains.**

Let k be a positive integer. It can be proved by induction that k distinct lines in a projective space intersect in at most $\binom{k}{2}$ points counted with multiplicity, and the maximal number is attained exactly when the k lines span a 2-plane.

Definition 2.3. *Let $k \geq 1$ be an integer. We call the union of k distinct lines which span a 2-plane a plane k-chain. Let $W \subset \mathbb{P}^N$ be the union of a finite number of lines. A plane k-chain in W is called maximal if it is not a subset of a plane k'-chain in W for some $k' > k$.*

Let $S \subset \mathbb{P}^5$ be a singular scroll with isolated singularities. There's a subset $W \subset S$ consisting of a finite number of rulings defined by

$$W = \bigcup l : l \text{ is a ruling passing through a singular point on } S. \tag{2.1}$$

By Zorn's lemma, W can be expressed as

$$W = \bigcup_{i=1}^{n} K_i : K_i \text{ is a maximal plane } k_i\text{-chain with } k_i \geq 2.$$

If two plane k-chains share more than one line, then they must lie on the same 2-plane. In particular, both of them can not be maximal. Therefore, for distinct maximal plane k-chains K_i and K_j in W, we have either $K_i \cap K_j = \emptyset$ or $K_i \cap K_j = l$ a single ruling. It follows that the number of singularities on S equals $\sum_{i=1}^{n} \binom{k_i}{2}$ since a plane k-chain contributes $\binom{k}{2}$ singularities.

Let $l_1, ..., l_k \subset S_{1,v}$ be k distinct rulings which span a subspace $P_{l_1,...,l_k} \subset \mathbb{P}^{D+1}$. The images of the rulings form a plane k-chain on S through projection if and only if $P_{l_1,...,l_k}$ is projected onto a 2-plane in $\mathbb{P}^5$. Parametrize the rulings as in (1.4) with $l_j = l_j(s_j)$, $j = 1, ..., k$. Then $P_{l_1,...,l_k}$ is spanned by the row vectors of the following $(k+2) \times (D+2)$ matrix

$$P(s_1, ..., s_k) = \begin{pmatrix} 1 & s_1 & 0 & 0 & ... & 0 \\ 1 & s_2 & 0 & 0 & ... & 0 \\ 0 & 0 & 1 & s_1 & ... & s_1^v \\ & & \vdots & & & \\ 0 & 0 & 1 & s_k & ... & s_k^v \end{pmatrix}. \tag{2.2}$$

The projection $S_{1,v} \to S$ is restricted from a linear map

$$\Lambda : \mathbb{P}^{D+1} \dashrightarrow \mathbb{P}^5.$$

Suppose Λ is represented by a $(D+2) \times 6$ matrix

$$\Lambda = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{pmatrix},$$

where $v_1, ..., v_6$ are vectors in $\mathbb{P}^{D+1}$. Then $P_{l_1,...,l_k}$ is projected onto a 2-plane if and only if the $(k+2) \times 6$ matrix

$$P(s_1, ..., s_k) \cdot \Lambda$$

has rank three.

2.2. **Control the number of singularities.** Let r be a non-negative integer. We introduce a method to find a projection Λ which maps $S_{1,v}$ to a singular scroll S with isolated singularities. The method allows us to control the number of singularities such that it is bounded below by r. For simplicity, we consider only the cases when the configuration $W \subset S$ defined in (2.1) consists of four disjoint maximal plane k-chains.

We start by picking distinct rulings on $S_{u,v}$ and produce four matrices P_1, P_2, P_3, and P_4 as in (2.2). Suppose P_i consists of k_i rulings. Note that

P_i contribute $\binom{k_i}{2}$ singularities if its rulings are mapped to a plane k_i-chain. Thus we also assume that

$$r = \binom{k_1}{2} + \binom{k_2}{2} + \binom{k_3}{2} + \binom{k_4}{2}. \tag{2.3}$$

Here we allow $k_i = 1$ which means that P_i consists of a single ruling and thus contributes no singularity.

Consider $\Lambda = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{pmatrix}$ as an unknown. Let P be the 5-plane spanned by $v_1, ..., v_6$. We are going to construct Λ satisfying

(1) $\mathrm{rk}\,(P_i \cdot \Lambda) = 3, \quad i = 1, 2, 3, 4.$
(2) $P^{\perp} \cap T(S_{1,v}) = \emptyset.$

Note that (1) makes the number of isolated singularities $\geq r$, while (2) confirms that no curve singularity occurs. We divide the construction into two steps:

Step 1. Find v_1, v_2, v_3 and v_4 to satisfy (1).

Consider each P_i as a linear map by multiplication from the left. We are trying to find independent vectors v_1, v_2, v_3 and v_4 such that for each P_i three of them are in the kernel while the remaining one isn't. The four vectors arranged in this way contribute exactly one rank to each $P_i \cdot \Lambda$. In the next step, v_5 and v_6 will be general vectors in $\mathbb{P}^{D+1}$ satisfying some open conditions. This contributes two additional ranks to each $P_i \cdot \Lambda$, which makes (1) true.

Under the standard parametrization for $S_{1,v} \subset \mathbb{P}^{D+1}$, the underlying vector space of $\mathbb{P}^{D+1}$ can be decomposed as $A \oplus B$ with A representing the first 2 coordinates and B representing the last $v + 1$ coordinates. With this decomposition, the matrix P in (2.2) can be decomposed into two Vandermonde matrices

$$P^A = \begin{pmatrix} 1 & s_1 \\ 1 & s_2 \end{pmatrix} \quad \text{and} \quad P^B = \begin{pmatrix} 1 & s_1 & \dots & s_1^v \\ & \vdots & & \\ 1 & s_k & \dots & s_k^v \end{pmatrix}.$$

Note that $\ker P = \ker P^B$. So we can search for the vectors from $\ker P^B$.

In our situation, we have four matrices ${P_1}^B$, ${P_2}^B$, ${P_3}^B$, ${P_4}^B$ which have four kernels $\ker {P_1}^B$, $\ker {P_2}^B$, $\ker {P_3}^B$, and $\ker {P_4}^B$, respectively. By the assumption $k_i \leq v$ and the fact that a Vandermonde matrix has full rank, each $\ker {P_i}^B$ is a codimension k_i subspace of B.

Now we want to pick $v_1, ..., v_4$ from B such that each $\ker P_i{}^B$ contains exactly three of the four vectors, i.e. we want

$$\left| \ker P_i{}^B \cap \{v_1, v_2, v_3, v_4\} \right| = 3, \quad \text{for } i = 1, 2, 3, 4. \tag{2.4}$$

One way to satisfy (2.4) is to pick v_i from

$$\left(\bigcap_{j \neq i} \ker P_j{}^B \right) - \ker P_i{}^B, \quad \text{for } i = 1, 2, 3, 4. \tag{2.5}$$

The sets in (2.5) are nonempty if and only if

$$\dim \left(\ker P_\alpha{}^B \cap \ker P_\beta{}^B \cap \ker P_\gamma{}^B \right) \geq 1$$

for all distinct $\alpha, \beta, \gamma \in \{1, 2, 3, 4\}$. This is equivalent to

$$k_\alpha + k_\beta + k_\gamma \leq v, \quad \text{for distinct } \alpha, \beta, \gamma \in \{1, 2, 3, 4\}. \tag{2.6}$$

So we have to include (2.6) as one of our assumptions.

Step 2. Adjust $v_1, ..., v_4$ and then pick v_5 and v_6 to satisfy (2).

Lemma 2.4. *Let $v_i{}^\perp$ be the hyperplane in $\mathbb{P}^{D+1}$ orthogonal to v_i. The four vectors $v_1, ..., v_4$ can be chosen generally such that $\bigcap_{i=1}^4 v_i{}^\perp$ intersects $T(S_{1,v})$ only in the directrix of $S_{1,v}$.*

Proof. Parametrize the rational normal curve $C = S_{1,v} \cap \mathbb{P}(B)$ by

$$\theta(s) = (0, 0, 1, s, ..., s^v).$$

Then the standard parametrization (1.3) can be written as

$$(1, s, 0, ..., 0) + t\theta(s).$$

Let a and b be the parameters for the tangent plane over each point. Then the tangent variety $T(S_{1,v})$ has the parametric equation

$$\begin{aligned} &(1, s, 0, ..., 0) + t\theta(s) + a\left[(0, 1, 0, ..., 0) + t\tfrac{d\theta}{ds}(s)\right] + b\theta(s) \\ &= (1, s + a, 0, ..., 0) + (t + b)\theta(s) + ta\tfrac{d\theta}{ds}(s). \end{aligned}$$

Each point on $T(S_{1,v})$ lying in $\bigcap_{i=1}^4 v_i{}^\perp$ is a common zero of the equations

$$(t + b)\left(\theta(s) \cdot v_i\right) + ta\left(\frac{d\theta}{ds}(s) \cdot v_i\right) = 0, \quad i = 1, 2, 3, 4. \tag{2.7}$$

By considering $(t + b)$ and ta as variables, (2.7) becomes a system of linear equations given by the matrix

$$\begin{pmatrix} \theta(s) \cdot v_1 & \theta(s) \cdot v_2 & \theta(s) \cdot v_3 & \theta(s) \cdot v_4 \\ \theta'(s) \cdot v_1 & \theta'(s) \cdot v_2 & \theta'(s) \cdot v_3 & \theta'(s) \cdot v_4 \end{pmatrix}.$$

The matrix fails to be of full rank exactly when s admits the existence of $\alpha, \beta \in \mathbb{C}$, $\alpha\beta \neq 0$, such that

$$(\alpha\theta(s) + \beta\theta'(s)) \cdot v_i = 0, \quad i = 1, 2, 3, 4. \tag{2.8}$$

Note that (2.8) has a solution if and only if $\bigcap_{i=1}^{4} v_i{}^{\perp}$ and the tangent variety $T(C)$ of C intersect each other.

One can choose v_2, v_3 and v_4 in general from (2.5) so that $\bigcap_{i=2}^{4} v_i{}^{\perp}$ is disjoint from C. This forces $\bigcap_{i=2}^{4} v_i{}^{\perp}$ to intersect $T(C)$ in either empty set or points. By the properties of a rational normal curve, the hyperplane orthogonal to a point on C contains no invariant subspace when one perturb the point. Hence, after necessary perturbation of the chosen rulings, one can choose v_1 from (2.5) such that $\left(\bigcap_{i=1}^{4} v_i{}^{\perp}\right) \cap T(C) = \emptyset$. As a result, the equations in (2.7) become independent, so the solutions are $t = b = 0$ or $a = 0, t = -b$. Both solutions form the directrix of $S_{1,v}$. □

With the above adjustment, we can pick v_5 and v_6 in general in $\mathbb{P}^{D+1}$ so that the $(D-5)$-plane $Q = v_1{}^{\perp} \cap \ldots \cap v_6{}^{\perp}$ has no intersection with $T(S_{1,v})$. Note that the projection defined by Λ is the same as the projection from Q. By Proposition 1.8, this projection produces a rational scroll with isolated singularities.

Proposition 2.5. *There exists a rational scroll in $\mathbb{P}^5$ of type $(1, v)$ with isolated singularities which has at least r singularities if there are four positive integers $k_1 \geq k_2 \geq k_3 \geq k_4$ satisfying (2.3) and (2.6):*

$$r = \binom{k_1}{2} + \binom{k_2}{2} + \binom{k_2}{2} + \binom{k_2}{2} \quad \textit{and} \quad k_1 + k_2 + k_3 \leq v.$$

Proposition 2.1 is obtained by expanding the binomial coefficients followed by a change of variables.

2.3. **Proof of Proposition 2.2.** In the following we exhibit an explicit example which can be manipulated by a computer algebra system over characteristic zero. The main program used in our work is Singular [DGPS15].

Consider $\mathbb{P}^{10}$ with homogeneous coordinate $\mathbf{x} = (x_0, \ldots, x_{10})$. We define the rational normal scroll $S_{1,8}$ by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ x_1 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \end{pmatrix}.$$

In order to project $S_{1,8}$ onto a rational scroll whose singular locus is zero dimensional and consists of at least eight singular points, we use the method introduced previously to construct a projection

$$\Lambda = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & 0 & 120 & -34 & -203 & 91 & 70 & -56 & 13 & -1 \\ 0 & 0 & 2880 & 5184 & -2372 & -2196 & 633 & 261 & -63 & -9 & 2 \\ 0 & 0 & 0 & 480 & 304 & -510 & -339 & 30 & 36 & 0 & -1 \\ 0 & 0 & 0 & 144 & 36 & -196 & -49 & 56 & 14 & -4 & -1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

Let $\mathbf{z} = (z_0, ..., z_5)$ be the coordinate for $\mathbb{P}^5$. Then the projection $\mathbb{P}^{10} \dashrightarrow \mathbb{P}^5$ defined by Λ can be explicite written by

$$\mathbf{z} = \mathbf{x} \cdot \Lambda.$$

Let S be the image of $S_{1,8}$ under the projection. Due to the limit of the author's computer, we check that S has eight singularities and smooth otherwise over the finite field of order 31. On the other hand, the double point formula implies that S has eight double points if the singular locus is isolated. Hence the singularity of S consists of eight double points over characteristic zero as required.

The generators of the ideal of S contain six cubics, so property (1) is confirmed. Properties (2) and (4) can be easily checked by examining the linear combinations of those cubics.

The final step is to varify property (3). Let $X \subset \mathbb{P}^5$ be a smooth cubic containing S. Let $F_1(S)$ and $F_1(X)$ denote the Fano variety of lines on S and X, respectively. Then it is equivalent to show that $F_1(S)$ deforms in $F_1(X)$ to the first order with dimension two.

Let $\mathbb{G}(1,5)$ be the grassmannian of lines in $\mathbb{P}^5$. Every element $\mathbf{b} \in \mathbb{G}(1,5)$ is parametrized by a 2×6 matrix

$$\begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} = \begin{pmatrix} b_{10} & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{20} & b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \end{pmatrix} \tag{2.9}$$

where $\mathbf{b}_1$ and $\mathbf{b}_2$ are two vectors which span the line $\mathbf{b}$.

Let $P_X = P_X(\mathbf{z})$ be the homogeneous polynomial defining X. Let V be the 6-dimensional linear space underlying $\mathbb{P}^5$. Consider P_X as a symmetric function defined on $V \oplus V \oplus V$. Then $F_1(X) \subset \mathbb{G}(1,5)$ is cut out by the four equations

$$P_X(\mathbf{b}_1, \mathbf{b}_1, \mathbf{b}_1),\ P_X(\mathbf{b}_1, \mathbf{b}_1, \mathbf{b}_2),\ P_X(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_2),\ P_X(\mathbf{b}_2, \mathbf{b}_2, \mathbf{b}_2). \tag{2.10}$$

Consider the Fano variety of lines on $S_{1,8}$ as a rational curve $\mathbb{P}^1 \subset \mathbb{G}(1,10)$ parametrized by

$$Q = \begin{pmatrix} r & s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r^8 & r^7 s & r^6 s^2 & r^5 s^3 & r^4 s^4 & r^3 s^5 & r^2 s^6 & r s^7 & s^8 \end{pmatrix}$$

where (r, s) is the homogeneous coordinate for $\mathbb{P}^1$. Then $F_1(S) \subset \mathbb{G}(1,5)$ is defined by the parametric equation

$$R = Q \cdot \Lambda.$$

Now consider a 2×6 matrix dR whose first row consists of arbitrary linear forms on $\mathbb{P}^1$ while the second row consists of arbitrary 8-forms. The coefficients of those forms introduce $2 \cdot 6 + 9 \cdot 6 = 66$ variables $c_1, ..., c_{66}$. Then an abstract first order deformation of $F_1(S)$ in $\mathbb{G}(1,5)$ is given by

$$R + dR.$$

Inserting $R + dR$ into (2.10) gives us four polynomials in r and s with coefficients in $c_1, ..., c_{66}$. The linear parts of the coefficients form a system of linear equations in $c_1, ..., c_{66}$ whose associated matrix has rank 53. Then the first order deformation of $F_1(S)$ in $F_1(X)$ appears as solutions of the system.

In addition to the 53 constraints contributed by the above linear equations, we also have

- 4 constraints from the GL(2) action on the coordinates (2.9).
- 3 constraints from the automorphism group of $\mathbb{P}^1$.
- 4 constraints from rescaling the four equations (2.10).

So $F_1(S)$ deforms in $F_1(X)$ to the first order with dimension $66 - 53 - 4 - 3 - 4 = 2$.

3. Special cubic fourfolds of discriminant 42

This section proves that a generic special cubic fourfold $X \in C_{42}$ has a unirational parametrization of odd degree and also that C_{42} is uniruled. We also provide a discussion in the end talking about the difficulty of generalizing our method to higher discriminants.

3.1. **The space of singular scrolls.** The Zariski closure of the locus for degree-9 scrolls forms a component $\mathcal{H}_9$ in the associated Hilbert scheme. Let $\mathcal{H}_9^8 \subset \mathcal{H}_9$ be the closure of the locus parametrizing scrolls with 8 isolated singularities. By Propositions 2.1 and Theorem 4.2 we have the estimate:

Corollary 3.1. *$\mathcal{H}_9^8$ has codimension at most 8 in $\mathcal{H}_9$.*

Note that $\mathcal{H}_9^8$ parametrizes non-reduced schemes by definition. In the following, we use an overline to specify an element $\overline{S} \in \mathcal{H}_9^8$ and denote by S its underlying reduced subscheme.

Let $U \subset |\mathcal{O}_{\mathbb{P}^5}(3)|$ be the locus parametrizing smooth cubic fourfolds. Define

$$\mathcal{Z} = \left\{(\overline{S}, X) \in \mathcal{H}_9^8 \times U : S \subset X\right\}.$$

By Proposition 2.2 there exists $(\overline{S}, X) \in \mathcal{Z}$ such that S has isolated singularities and X is smooth.

The right projection $p_2 : \mathcal{Z} \to U$ factors through U_{42}, the preimage of C_{42} in U. Indeed, by definition S is the image of a rational normal scroll $F \subset \mathbb{P}^{10}$ through a projection. Let $\epsilon : F \to X$ be the composition of the projection followed by the inclusion into X. Let $[S]_X{}^2$ is the self-intersection of S in X. Then the number of singularities $D_{S\subset X} = 8$ on S satisfies the double point formula [Ful98, Th. 9.3]:

$$D_{S\subset X} = \frac{1}{2}\left([S]_X{}^2 - \epsilon^* c_2(T_X) + c_1(T_F)\cdot \epsilon^* c_1(T_X) - c_1(T_F)^2 + c_2(T_F)\right).$$

By using this formula on can get $[S]_X{}^2 = 41$. Let h_X be the hyperplane class of X. Then the intersection table for X is

	h_X^2	S
h_X^2	3	9
S	9	41.

So X has discriminant $3 \cdot 41 - 9^2 = 42$.

3.2. **Odd degree unirational parametrizations.**

Theorem 3.2. *Consider the diagram*

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & \\ \mathcal{H}_9^8 & & U_{42} \arrow[r] & C_{42} \end{tikzcd}$$

(1) *$\mathcal{Z}$ dominates U_{42}. Therefore a general $X \in C_{42}$ contains a degree-9 rational scroll with 8 isolated singularities and smooth otherwise.*
(2) *C_{42} is uniruled.*

Proof. Let $(\overline{S}, X) \in \mathcal{Z}$ be a pair satisfying Proposition 2.2. Then

$$h^0(\mathbb{P}^5, \mathcal{I}_S(3)) = 6.$$

On the other hand, the short exact sequence

$$0 \to \mathcal{I}_S(3) \to \mathcal{O}_{\mathbb{P}^5}(3) \to \mathcal{O}_S(3) \to 0$$

implies that

$$h^0(\mathbb{P}^5, \mathcal{I}_S(3)) \geq h^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)) - h^0(S, \mathcal{O}_S(3)). \tag{3.1}$$

Let $F \subset \mathbb{P}^{10}$ be the preimage scroll of S. Then $H^0(S, \mathcal{O}_S(3))$ consists of the sections in $H^0(F, \mathcal{O}_F(3))$ which cannot distinguish the preimage of a singular point. We have $h^0(F, \mathcal{O}_F(3)) = 58$ by Lemma 1.1, so $h^0(S, \mathcal{O}_S(3)) = 58 - 8 = 50$. So the right hand side of (3.1) equals $56 - 50 = 6$. Thus $h^0(\mathbb{P}^5, \mathcal{I}_S(3))$ attains a minimum.

The left projection $p_1 : \mathcal{Z} \to \mathcal{H}_9^8$ has fiber $\mathbb{P}H^0(\mathbb{P}^5, \mathcal{I}_S(3))$ over all $\overline{S} \in \mathcal{H}_9^8$. Because the fiber dimension is an upper-semicontinuous function, there is an open subset $V \subset \mathcal{H}_9^8$ containing $\overline{S}$ such that $\mathcal{Z}$ is a $\mathbb{P}^5$-bundle over V. We have $\dim \mathcal{H}_9 = 59$ by Proposition 4.1. Hence $\dim \mathcal{H}_9^8 \geq 59 - 8 = 51$ by Theorem 4.2. Thus $\mathcal{Z}$ has dimension at least $51 + 5 = 56$ in a neighborhood of $(\overline{S}, X)$.

By Proposition 2.2 (3), $\mathcal{Z}$ has fiber dimension at most 2 over an open subset of $p_2(\mathcal{Z})$ which contains X. Hence $p_2(\mathcal{Z})$ has dimension at least $56 - 2 = 54$ in a neighborhood of X. On the other hand, U_{42} is an irreducible divisor in U. In particular, U_{42} has dimension 54. So $\mathcal{Z}$ must dominate U_{42}.

Next we prove the uniruledness of C_{42}.

We already know that $\mathcal{Z}$ has an open dense subset $\mathcal{Z}^\circ$ isomorphic to a $\mathbb{P}^5$-bundle over $V \subset \mathcal{H}_9^8$. If we can prove that the composition $\mathcal{Z}^\circ \xrightarrow{p_2} U_{42} \to C_{42}$ does not factor through this bundle map, then the proof is done.

Let $(\overline{S}, X) \in \mathcal{Z}^\circ$ be the pair as before. By Proposition 2.2, S is also contained in a singular cubic Y. Assume that the map $\mathcal{Z}^\circ \to C_{42}$ does factor through the bundle map instead. Then all of the cubics in $\mathbb{P}H^0(\mathbb{P}^5, \mathcal{I}_S(3))$ would be in the same $\mathbb{P}\mathrm{GL}(6)$-orbit. In particular, the smooth cubic X and the singular cubic Y would be isomorphic, but this is impossible. □

Proposition 3.3. [Has16, Prop. 38] [HT01, Prop. 7.4] *Let X be a cubic fourfold and $S \subset X$ be a rational surface. Suppose S has isolated singularities and smooth normalization, with invariants $D = \deg S$, section genus g_H, and self-intersection $\langle S, S\rangle_X$. If*

$$\varrho = \varrho(S, X) := \frac{D(D-2)}{2} + (2 - 2g_H) - \frac{\langle S, S\rangle_X}{2} > 0, \tag{3.2}$$

then X admits a unirational parametrization $\rho : \mathbb{P}^4 \dashrightarrow X$ of degree ϱ.

Corollary 3.4. *A general $X \in C_{42}$ has an unirational parametrization of degree 13.*

Proof. By Theorem 3.2 (1), a general cubic fourfold $X \in \mathcal{C}_{42}$ contains a degree-9 scroll S having 8 isolated singularties, with $\langle S, S\rangle_X = 41$ and $g_H = 0$. Thus $\varrho = \frac{9\cdot 7}{2} + 2 - \frac{41}{2} = 13$ by Proposition 3.3. □

3.3. **Problems in higher discriminants.** Let $\Delta \subset \Sigma^{[2]}$ denote the divisor parametrizing non-reduced subschemes. Recall that it is a $\mathbb{P}^1$-bundle over Σ. Its fibers correspond to smooth rational curves of degree $2n+1$ in $F_1(X)$, where the polarization on $F_1(X)$ is induced from $\mathbb{G}(1,5)$. Each rational scroll $S \subset X$ induced by these rational curves has the intersection product

$$\begin{array}{c|cc} & h_X^2 & S \\ \hline h_X^2 & 3 & 2n+1 \\ S & 2n+1 & 2n^2+2n+1, \end{array}$$

where h_X is the hyperplane section class of X. One can compute by the double point formula that S has $n(n-2)$ singularities provided that they are all isolated. [HT01, Prop. 7.2]

In order to obtain an odd degree unirational parametrization for a generic member in $\mathcal{C}_d$ by Proposition 3.3, we need the existence of a degree $2n+1$ scroll $S \subset \mathbb{P}^5$ with isolated singularities which has $n(n-2)$ singularities and is contained in a cubic fourfold X. We also need an estimate on the dimension of the associated Hilbert scheme $\mathcal{H}_{2n+1}^{n(n-2)}$ which contains S.

Section 2 builds up a method to find such S, but the existence of a cubic fourfold X containing S requires examination with a computer. This works well with $n = 4$ because in this case a generic such S is contained in a cubic hypersurface. However, the same phenomenon may fail when $n \geq 5$. Indeed, the Hilbert scheme $\mathcal{H}_{2n+1}^{n(n-2)}$ of degree $2n+1$ scrolls with $n(n-2)$ singularities satisfies $\dim \mathcal{H}_{2n+1}^{n(n-2)} \geq -n^2 + 14n + 11$ by Theorem 4.2 and Proposition 4.1. When $5 \leq n \leq 8$, $\dim \mathcal{H}_{2n+1}^{n(n-2)} \geq 55$ the dimension of cubic hypersurfaces in $\mathbb{P}^5$, so a generic $S \in \mathcal{H}_{2n+1}^{n(n-2)}$ is not in a cubic fourfold. We don't know what happens when $n \geq 9$, but working in this range involves tedious trial and error.

Question. Assume $n \geq 2$. Let $S \subset \mathbb{P}^5$ be a degree $2n+1$ rational scroll which has $n(n-2)$ isolated singularities and smooth otherwise. When is S contained in a cubic fourfold?

4. The Hilbert scheme of rational scrolls

Let $N \geq 3$ be an integer. The Hilbert polynomial P_S for a degree D smooth surface $S \subset \mathbb{P}^N$ has the following form

$$P_S(x) = \frac{1}{2}Dx^2 + \left(\frac{1}{2}D + 1 - \pi\right)x + 1 + p_a,$$

where π is the genus of a generic hyperplane section and p_a is the arithmetic genus of S. [Har77, V, Ex 1.2]

We are interested in the case when S is a rational scroll. In this case $\pi = p_a = 0$, so

$$P_S(x) = \frac{D}{2}x^2 + \left(\frac{D}{2} + 1\right)x + 1.$$

Every smooth surface sharing the same Hilbert polynomial has $\pi = 0$ and $p_a = 0$ also and thus is rational. We denote by $\mathrm{Hilb}_{P_S}(\mathbb{P}^N)$ the Hilbert scheme of subschemes in $\mathbb{P}^N$ with Hilbert polynomial P_S.

The closure of the locus parametrizing degree D scrolls forms a component $\mathcal{H}_D \subset \mathrm{Hilb}_{P_S}(\mathbb{P}^N)$. We study this space by stratifying it according to the types of the scrolls. Recall that, by fixing a rational normal scroll $S_{u,v} \subset \mathbb{P}^{D+1}$ where $D = u + v$, a rational scroll $S \subset \mathbb{P}^N$ of type (u, v) is either $S_{u,v}$ itself or the image of $S_{u,v}$ projected from a disjoint $(D - N)$-plane. We define $\mathcal{H}_{u,v} \subset \mathcal{H}_D$ as the closure of the subset consisting of smooth rational scrolls of type (u, v). In this section, we will first show that

Proposition 4.1. *Assume $D + 1 \geq N \geq 3$.*

(1) *$\mathcal{H}_D$ is generically smooth of dimension $(N + 1)(D + 2) - 7$.*
(2) *$\mathcal{H}_{u,v}$ is unirational of dimension $(D + 2)N + 2u - 4 - \delta_{u,v}$,*

where $\delta_{u,v}$ is the Kronecker delta. We also have

(3) *$\mathcal{H}_{u,v} \subset \mathcal{H}_{u+k,v-k}$ for $0 \leq 2k < v - u$, and $\mathcal{H}_{\lfloor \frac{D}{2} \rfloor, \lceil \frac{D}{2} \rceil} = \mathcal{H}_D$.*

When $D + 1 = N$, a generic element of $\mathcal{H}_{u,v}$ is projectively equivalent to a fixed rational normal scroll $S_{u,v} \subset \mathbb{P}^{D+1}$. In this case $\mathcal{H}_{u,v}$ is birational to $\mathbb{P}\mathrm{GL}(D + 2)$ quotient by the stablizer of $S_{u,v}$.

When $D \geq N$, a generic element in $\mathcal{H}_{u,v}$ is the projection of $S_{u,v}$ from a $(D-N)$-plane. Note that $\mathcal{H}_{u,v}$ also records the scrolls equipped with embedded points along their singular loci. Such element occurs when the $(D-N)$-plane contacts the secant variety of $S_{u,v}$. We denote by $\mathcal{H}^r_{u,v} \subset \mathcal{H}_{u,v}$ the closure of the subset parametrizing the schemes such that the singular locus of each of the underlying varieties consists of $\geq r$ isolated singularities. Let $\mathcal{H}^r_D \subset \mathcal{H}_D$ denote the union of $\mathcal{H}^r_{u,v}$ through all possible types.

The main goal of this section is to prove the following theorem

Theorem 4.2. *Assume $D \geq N \geq 5$, and assume the existence of a degree D rational scroll with isolated singularities in $\mathbb{P}^N$ which has at least r singularities. Suppose $rN \leq (D+2)^2 - 1$, then $\mathcal{H}_D^r$ has codimension at most $r(N-4)$ in $\mathcal{H}_D$. Especially when $r = 1$, $\mathcal{H}_D^1$ is unirational of codimension exactly $N-4$.*

4.1. **The component of rational scrolls.** Here we give a general picture of the component $\mathcal{H}_D$ and also prove Proposition 4.1. Note that Proposition 4.1 (1) follows immediately from Lemma 1.5.

As mentioned before, $\mathcal{H}_{u,v}$ is birational to $\mathbb{P}\mathrm{GL}(D+2)$ when $D+1 = N$. In order to study the case of $D \geq N$, we introduce the *projective Stiefel variety*.

Definition 4.3. *Let $V_{N+1}(\mathbb{C}^{D+2}) = \mathrm{GL(D+2)/GL(D-N+1)}$ be the homogeneous space of $(N+1)$-frames in $\mathbb{C}^{D+2}$. The group $\mathbb{C}^*$ acts on $V_{N+1}(\mathbb{C}^{D+2})$ by rescaling, which induces a geometric quotient $\mathbb{V}(N, D+1)$ that we call a projective Stiefel variety.*

$\mathbb{V}(N, D+1)$ has a fiber structure over $\mathbb{G}(N, D+1)$:

$$\begin{array}{ccc} \mathbb{P}\mathrm{GL}(N+1) & \hookrightarrow & \mathbb{V}(N, D+1) \\ & & \downarrow p \\ & & \mathbb{G}(N, D+1) \end{array} .$$

An element $\Lambda \in \mathbb{V}(N, D+1)$ over $P \in \mathbb{G}(N, D+1)$ can be expressed as a $(D+2) \times (N+1)$-matrix

$$\Lambda = \begin{pmatrix} v_1 & v_2 & \dots & v_{N+1} \end{pmatrix}_{(D+2)\times(N+1)}$$

up to rescaling, where $v_1, ..., v_{N+1}$ are column vectors which form a basis of the underlying vector space of P. In particular, each $\Lambda \in \mathbb{V}(N, D+1)$ naturally defines a projection $\cdot\Lambda : \mathbb{P}^{D+1} \dashrightarrow \mathbb{P}^N$ by multiplying the coordinates from the right.

Let $S_{u,v} \subset \mathbb{P}^{D+1}$ be the rational normal scroll given by the standard parametrization (1.3). When $D \geq N$, every rational scroll in $\mathcal{H}_{u,v}$ is the image of $S_{u,v}$ under the projection defined by some $\Lambda \in \mathbb{V}(N, D+1)$. So there is a dominant rational map

$$\begin{array}{cccc} \pi = \pi(S_{u,v}): & \mathbb{V}(N, D+1) & \dashrightarrow & \mathcal{H}_{u,v} \\ & \Lambda & \longmapsto & S_{u,v} \cdot \Lambda, \end{array} \tag{4.1}$$

where $S_{u,v} \cdot \Lambda$ is the rational scroll given by the parametric equation

$$\begin{array}{ccc} \mathbb{C}^2 & \longrightarrow & \mathbb{P}^N \\ (s,t) & \longmapsto & (1, s, ..., s^u, t, st, ..., s^v t) \cdot \Lambda. \end{array}$$

Proof of Proposition 4.1 (2). Both $\mathbb{PGL}(D+2)$ and $\mathbb{V}(N, D+1)$ are rational quasi-projective varieties, so $\mathcal{H}_{u,v}$ is unirational either when $D+1 = N$ or $D \geq N$ by the above construction. The formula for the dimension of $\mathcal{H}_{u,v}$ holds by [Cos06, Lemma 2.6]. □

Proof of Proposition 4.1 (3). By Lemma 1.3, there exists an embedded deformation $\mathcal{S}$ in $\mathbb{P}^{D+1}$ over the dual numbers $D_t = \frac{\mathbb{C}[t]}{(t^2)}$ with $\mathcal{S}_0 \cong S_{u,v}$ and $\mathcal{S}_t \cong S_{u+k,v-k}$ for $t \neq 0$. For every rational scroll $S \in \mathcal{H}_{u,v}$, we can find a $\Lambda \in \mathbb{V}(N, D+1)$ such that $S = S_{u,v} \cdot \Lambda$. Then $\mathcal{S} \cdot \Lambda$ defines an infinitesimal deformation of S to a rational scroll of type $(u+k, v-k)$, which forces the inclusion $\mathcal{H}_{u,v} \subset \mathcal{H}_{u+k,v-k}$ to hold.

When $(u, v) = (\lfloor \frac{D}{2} \rfloor, \lceil \frac{D}{2} \rceil)$, i.e. when $u = v$ or $u = v-1$, we have $\dim \mathcal{H}_D = \dim \mathcal{H}_{u,v} = (N+1)(D+2) - 7$ by Proposition 4.1 (1) and (2). Because $\mathcal{H}_D = \bigcup_{u+v=D} \mathcal{H}_{u,v}$, we must have $\mathcal{H}_{\lfloor \frac{D}{2} \rfloor, \lceil \frac{D}{2} \rceil} = \mathcal{H}_D$. □

4.2. **Projections that produce one singularity.** We are ready to study the locus in $\mathcal{H}_D$ which parametrizes singular scrolls. Assume $D \geq N \geq 5$. Let us start from studying the projections that produce one singularity.

Notations & Facts. Let K and L be any linear subspaces of $\mathbb{P}^{D+1}$.

(1) We use the same symbol to denote a projective space and its underlying vector space. The dimension always means the projective dimension.
(2) Assume $K \subset L$, we write $K^{\perp L}$ for the orthogonal complement of K in L. When $L = \mathbb{P}^{D+1}$, we write $K^{\perp}$ instead of $K^{\perp \mathbb{P}^{D+1}}$.
(3) $K + L$ means the space spanned by K and L. We write it as $K \oplus L$ if $K \cap L = \{0\}$, and write it as $K \oplus_{\perp} L$ if K and L are orthogonal to each other.

The following two relations can be derived by linear algebra.

$$(K \cap L)^{\perp} = K^{\perp} + L^{\perp}. \tag{4.2}$$

$$(K \cap L)^{\perp K} = (K \cap L)^{\perp} \cap K. \tag{4.3}$$

Definition 4.4. *Let l and l' be a pair of distinct rulings on $S_{u,v}$, and let $P_{l,l'}$ be the 3-plane spanned by them. We define $\sigma(l, l')$ to be a subvariety of $\mathbb{G}(N, D+1)$ by*

$$\sigma(l, l') = \left\{ P \in \mathbb{G}(N, D+1) \;:\; \dim(P \cap P_{l,l'}^{\perp}) \geq N-3 \right\}.$$

Lemma 4.5. *Let $p : \mathbb{V}(N, D+1) \to \mathbb{G}(N, D+1)$ be the bundle map. Then $p^{-1}(\,\sigma(l, l')\,) \subset \mathbb{V}(N, D+1)$ consists of the projections which produce singularities by making l and l' intersect.*

Proof. Let $P \in \mathbb{G}(N, D+1)$ and $\Lambda \in p^{-1}(P)$ be arbitrary. The target space of the projection map $\cdot\Lambda$ is actually P. Let $L \subset \mathbb{P}^{D+1}$ be any linear subspace, then the image $L \cdot \Lambda$ is identical to $(P^{\perp} + L) \cap P$. On the other hand, (4.2) and (4.3) implies that $(P \cap L^{\perp})^{\perp P} = (P \cap L^{\perp})^{\perp} \cap P = (P^{\perp} + L) \cap P$. Therefore,

$$\begin{aligned} N - 1 &= \dim P - 1 = \dim(P \cap L^{\perp}) + \dim(P \cap L^{\perp})^{\perp P} \\ &= \dim(P \cap L^{\perp}) + \dim\left((P^{\perp} + L) \cap P\right) = \dim(P \cap L^{\perp}) + \dim\left(L \cdot \Lambda\right). \end{aligned}$$

With $L = P_{l,l'}$, the equation implies that

$$\dim(P \cap P_{l,l'}^{\perp}) \geq N - 3 \quad \Leftrightarrow \quad \dim\left(P_{l,l'} \cdot \Lambda\right) \leq 2.$$

It follows that

$$p^{-1}(\,\sigma(l, l')\,) = \{\Lambda \in \mathbb{V}(N, D+1) \,:\, \dim(P_{l,l'} \cdot \Lambda) \leq 2\}.$$

The image $P_{l,l'} \cdot \Lambda \subset \mathbb{P}^N$ lies in a plane if and only if l and l' intersect each other after the projection $\cdot\Lambda : \mathbb{P}^{D+1} \dashrightarrow \mathbb{P}^N$. As a consequence, every $\Lambda \in p^{-1}(\,\sigma(l, l')\,)$ defines a projection which produces a singularity by making l and l' intersect. □

4.3. **The geometry of the variety $\sigma(l, l')$.** The properties of the singular scroll locus that we are interested in are the unirationality and the dimension. As a preliminary, we describe here the geometry of the variety $\sigma(l, l')$, which implies immediately the rationality of $\sigma(l, l')$ and also allows us to find its dimension easily.

Instead of studying $\sigma(l, l')$ alone, the geometry would be more apparent if we consider generally the linear subspaces in $\mathbb{P}^{D+1}$ which satisfies a certain intersectional condition. Fix a $(D-3)$-plane $L \subset \mathbb{P}^{D+1}$. For every $j \geq 0$, we define

$$\sigma_j(L) = \{\, P \in \mathbb{G}(N, D+1) : \dim(P \cap L) \geq N - 4 + j \,\}. \tag{4.4}$$

For example, $\sigma_0(L) = \mathbb{G}(N, D+1)$, and $\sigma_1(P_{l,l'}^{\perp}) = \sigma(l, l')$. Note that $P \subset L$ or $L \subset P$ if $j \geq \min(4\,,\, D - N + 1)$ in (4.4), so we have

$$\begin{aligned} &\sigma_j(L) \supsetneq \sigma_{j+1}(L) \quad \text{if} \quad 0 \leq j < \min(4\,,\, D-N+1)\,, \\ &\sigma_j(L) = \sigma_{j+1}(L) \quad \text{if} \quad j \geq \min(4\,,\, D-N+1)\,. \end{aligned}$$

Define $\sigma_j^{\circ}(L) = \{P \in \mathbb{G}(N, D+1) \,:\, \dim(P \cap L) = N - 4 + j\}$, then

$$\begin{aligned} &\sigma_j^{\circ}(L) = \sigma_j(L) - \sigma_{j+1}(L) \quad \text{if} \quad 0 \leq j < \min(4\,,\, D-N+1)\,, \\ &\sigma_j^{\circ}(L) = \sigma_j(L) \qquad\qquad\quad\;\; \text{if} \quad j = \min(4\,,\, D-N+1)\,. \end{aligned}$$

Lemma 4.6. *Assume* $1 \le j < \min(4\,, D-N+1)$*, then* $\sigma_j(L)$ *is singular along* $\sigma_{j+1}(L)$ *and smooth otherwise. The singularity can be resolved by a* $\mathbb{G}(3-j\,, D-N+4-j)$*-bundle over* $\mathbb{G}(N-4+j\,, D-3)$*. Especially,* $\sigma_j(L)$ *is rational with codimension* $j(N-3+j)$ *in* $\mathbb{G}(N, D+1)$*.*

Proof. We define $\mathbf{G}_j(L)$ to be the fiber bundle

$$\begin{array}{ccc} \mathbb{G}(3-j\,, D-N+4-j) & \hookrightarrow & \mathbf{G}_j(L) \\ & & \downarrow \\ & & \mathbb{G}(N-4+j\,, L) \end{array}$$

by taking $\mathbb{G}(3-j\,, Q^{\perp})$ as the fiber over $Q \in \mathbb{G}(N-4+j\,, L)$. Apparently $\mathbf{G}_j(L)$ is smooth and rational. We denote an element of $\mathbf{G}_j(L)$ as (Q, R), where Q belongs to the base and R belongs to the fiber over Q.

In the following, we will construct a birational morphism from $\mathbf{G}_j(L)$ to $\sigma_j(L)$, which determines the rationality and the codimension immediately. Then we will study the singular locus by analyzing the tangent cone to $\sigma_j(L)$ at a point on $\sigma_{j+1}(L)$.

Step 1. A birational morphism from $\mathbf{G}_j(L)$ *to* $\sigma_j(L)$*.*

Every $P \in \sigma_j^{\circ}(L)$ can be decomposed as $P = (P \cap L) \oplus_{\perp} (P \cap L)^{\perp P}$. Because $P \cap L \in \mathbb{G}(N-4+j\,, L)$ and $(P \cap L)^{\perp P}$ is a $(3-j)$-plane in $(P \cap L)^{\perp}$, this induces a morphism

$$\begin{array}{rccc} \iota: & \sigma_j^{\circ}(L) & \longrightarrow & \mathbf{G}_j(L) \\ & P & \longmapsto & \Big(P \cap L\,, (P \cap L)^{\perp P}\Big). \end{array}$$

On the other hand, $Q \oplus_{\perp} R \in \sigma_j(L)$ for every $(Q, R) \in \mathbf{G}_j(L)$ since $\dim(Q \cap L) = N-4+j$ by definition. Thus there is a morphism

$$\begin{array}{rccc} \epsilon: & \mathbf{G}_j(L) & \longrightarrow & \sigma_j(L) \\ & (Q,R) & \longmapsto & Q \oplus_{\perp} R. \end{array} \tag{4.5}$$

Clearly, the composition $\epsilon \circ \iota$ is the same as the inclusion $\sigma_j^{\circ}(L) \subset \sigma_j(L)$. Therefore ϵ is a birational morphism.

The smoothness and rationality of $\mathbf{G}_j(L)$ implies that $\sigma_j^{\circ}(L)$ is smooth and that $\sigma_j(L)$ is rational. Moreover,

$$\begin{aligned} &\dim \sigma_j(L) = \dim \mathbf{G}_j(L) \\ &= (4-j)(D-N+1) + (N-3+j)(D-N+1-j) \\ &= (N+1)(D-N+1) - j(N-3+j) \\ &= \dim \mathbb{G}(N, D+1) - j(N-3+j). \end{aligned}$$

Hence $\sigma_j(L)$ has codimension $j(N-3+j)$ in $\mathbb{G}(N, D+1)$.

Step 2. The tangent cones to $\sigma_j(L)$*.*

Choose any $P \in \sigma_j(L)$ and fix a $\phi \in T_P\mathbb{G}(N, D+1) \cong \operatorname{Hom}(P, P^\perp)$. Let $T_P\sigma_j(L)$ be the tangent cone to $\sigma_j(L)$ at P. By definition, $\phi \in T_P\sigma_j(L)$ if and only if the condition $\dim(P \cap L) \geq N - 4 + j$ is kept when P moves infinitesimally in the direction of ϕ, which is equivalent to the condition that $P \cap L$ has a subspace Q of dimension $N - 4 + j$ such that $\phi(Q) \subset L$.

Consider the decomposition

$$P^\perp = (P^\perp \cap L) \oplus_\perp (P^\perp \cap L)^{\perp P^\perp}.$$

Define

$$\Gamma : \operatorname{Hom}(P, P^\perp) \to \operatorname{Hom}\left(P \cap L, (P^\perp \cap L)^{\perp P^\perp}\right)$$

to be the composition of the restriction to $P \cap L$ followed by the right projection of the above decomposition.

For any subspace $Q \subset P \cap L$, $\phi(Q) \subset L$ if and only if $\phi(Q) \subset P^\perp \cap L$, if and only if $Q \subset \ker \Gamma(\phi)$. So L has a subspace Q of dimension $N - 4 + j$ such that $\phi(Q) \subset L$ if and only if the (projective) dimension of $\ker \Gamma(\phi)$ is at least $N - 4 + j$. Therefore,

$$T_P\sigma_j(L) = \{\phi \in \operatorname{Hom}(P, P^\perp) : \dim(\ker \Gamma(\phi)) \geq N - 4 + j\}. \tag{4.6}$$

Note that $\sigma_j(L)$ is the disjoint union of $\sigma^\circ_{j+k}(L)$ for all k satisfying

$$0 \leq k \leq \min(4\,,\ D - N + 1) - j.$$

Assume $P \in \sigma^\circ_{j+k}(L)$, i.e. $\dim(P \cap L) = N - 4 + j + k$, then (4.6) is equivalent to

$$T_P\sigma_j(L) = \{\phi \in \operatorname{Hom}(P, P^\perp) : \operatorname{rk} \Gamma(\phi) \leq k\}. \tag{4.7}$$

When $k = 0$, the constraint becomes $\operatorname{rk} \Gamma(\phi) = 0$, so $T_P\sigma_j(L) = \ker \Gamma$ is a vector space. This reflects the fact that $\sigma_j(L)$ is smooth on $\sigma^\circ_j(L)$ for all j. On the other hand, from the inequality

$$\dim(P \cap L) + \dim(P^\perp \cap L) \leq \dim(L) - 1,$$

we get

$$\begin{aligned} &\dim(P^\perp \cap L) \leq \dim(L) - \dim(P \cap L) - 1 \\ &= (D - 3) - (N - 4 + j + k) - 1 = D - N - j - k. \end{aligned}$$

It follows that

$$\begin{aligned} &\dim\left((P^\perp \cap L)^{\perp P^\perp}\right) = \dim(P^\perp) - \dim(P^\perp \cap L) - 1 \\ &\geq (D - N) - (D - N - j - k) - 1 = j + k - 1. \end{aligned}$$

So $\dim\left((P^\perp \cap L)^{\perp P^\perp}\right) \geq j + k - 1 \geq k$ once $k \geq 1$. Under this condition, the linear combination of members of rank k in $\operatorname{Hom}\left(P \cap L, (P^\perp \cap L)^{\perp P^\perp}\right)$ can have rank exceeding k. So $T_P\sigma_j(L)$ can not be a vector space, thus P is a singularity of $\sigma_j(L)$. $\square$

Recall that $\sigma(l, l') = \sigma_1(P^{\perp}_{l,l'})$, so Lemma 4.6 implies that

Corollary 4.7. *$\sigma(l, l')$ is rational with codimension $N - 2$ in $\mathbb{G}(N, D+1)$.*

4.4. **Families of the projections.** The singularities we have studied are those produced from the intersection of a fixed pair of distinct rulings. Now we are going to make use of the variety $\sigma(l, l')$ to control multiple singularities.

Let $\mathbb{P}^{1[2]}$ be the Hilbert scheme of two points on $\mathbb{P}^1$ and $U \subset \mathbb{P}^{1[2]}$ be the open subset parametrizing reduced subschemes. On the rational normal scroll $S_{u,v}$, the set of r pairs of distinct rulings

$$\{(l_1 + l_1', ..., l_r + l_r') : l_i \neq l_i' \ \forall i\}$$

is parametrized by $U^{\times r}$.

Let Σ_r be a subset of $U^{\times r} \times \mathbb{G}(N, D+1)$ defined by

$$\Sigma_r = \left\{(l_1 + l_1', ..., l_r + l_r', P) \in U^{\times r} \times \mathbb{G}(N, D+1) \ : \ P \in \bigcap_{i=1}^{r} \sigma(l_i, l_i')\right\}.$$

Let p_1 be the left projection and p_2 the right projection. Then there is a diagram

$$\begin{array}{ccccc} \bigcap_{i=1}^r \sigma(l_i, l_i') & \subset & \Sigma_r & \xrightarrow{p_2} & \mathbb{G}(N, D+1) \\ \downarrow & & p_1 \downarrow & & \\ (l_1 + l_1', ..., l_r + l_r') & \in & \mathrm{U}^{\times r}. & & \end{array}$$

By Lemma 4.5, the image $p_2(\Sigma_r)$ consists of the N-planes such that the projections to them produce at least r singularities. By the diagram above and Corollary 4.7, the codimension of Σ_r in $U^{\times r} \times \mathbb{G}(N, D+1)$ is at most $r(N-4)$. When $r = 1$, Σ_1 is rational with codimension exactly $N - 4$.

Our goal is to compute the dimension of $p_2(\Sigma_r)$, so we care about whether p_2 is generically finite onto its image or not. It turns out that the condition below is sufficient (See Lemma 4.9)

(4.8) There exists a rational scroll with isolated singularities $S \subset \mathbb{P}^N$ of type (u, v) which has at least r singularities.

By considering S as the projection of $S_{u,v}$ from $P^{\perp}$ for some N-plane P, we can apply Corollary 1.7 to translate (4.8) into the equivalent statement:

(4.8') There exists an N-plane P such that $P^{\perp}$ intersects $S(S_{u,v})$ in $\geq r$ points away from $T(S_{u,v}) \cup Z_2$.

Proposition 4.8. *(4.8) holds for $r \leq D - N + 1$.*

Proof. By [CJ96, Prop. 2.2] and [Har95, Example 19.10], $\deg(S(S_{u,v})) = \binom{D-2}{2}$. Since $\dim(S(S_{u,v})) = 5$ and $T(S_{u,v}) \cup Z_2$ form a proper closed subvariety of $S(S_{u,v})$, we can use Bertini's theorem to choose a $(D-4)$-plane R which intersects $S(S_{u,v})$ in $\binom{D-2}{2}$ points outside $T(S_{u,v}) \cup Z_2$. It is easy to check that $\binom{D-2}{2} \geq D - N + 1$. Thus we can choose $D - N + 1$ of the intersection points to span a $(D-N)$-plane $Q \subset R$. Then $P = Q^{\perp}$ satisfies the hypothesis. $\square$

Unfortunately, Proposition 4.8 doesn't cover the case $D = 9$, $N = 5$ and $r = 8$ in our proof of the unirationality of discriminant 42 cubic fourfolds. In the following, we estimate the dimension of $p_2(\Sigma_r)$ under the assumption (4.8) and leave the construction of examples to Section 2.

Lemma 4.9. *Suppose (4.8) holds. Then $p_2(\Sigma_r)$ has codimension $\leq r(N-4)$ in $\mathbb{G}(N, D+1)$. When $r = 1$, $p_2(\Sigma_1)$ is rational of codimension exactly $N-4$.*

Proof. Let l and l' be distinct rulings on $S_{u,v}$. We write $P_{l,l'}$ for the 3-plane spanned by them. Note that $S(S_{u,v}) = \overline{\bigcup_{l \neq l'} P_{l,l'}}$. Let P be an N-plane satisfying (4.8'). Then there exists r pairs of distinct rulings $(l_1, l_1'), \ldots, (l_r, l_r')$ such that $P^{\perp}$ and $P_{l_i, l_i'}$ intersect in exactly one point for each pair (l_i, l_i'). This implies that $\dim(P^{\perp} + P_{l_i,l_i'}) \leq D - N + 3$ for all i, which is equivalent to $\dim\left(P \cap P^{\perp}_{l_i,l_i'}\right) \geq N - 3$ for all i by (4.2). Hence $P \in \bigcap_{i=1}^{r} \sigma(l_i, l_i')$, i.e. P belongs to the image of p_2.

Suppose $P^{\perp}$ intersects $S(S_{u,v})$ in m points, then the $(l_1 + l_1', \ldots, l_r + l_r')$ in the preimage of P is unique up to the choices of r from m pairs, the reordering of the r pairs, and the transpositions of the rulings in a pair. Hence p_2 is generically finite with $\deg p_2 = \binom{m}{r} \cdot r!$. In particular, Σ_r and $p_2(\Sigma_r)$ are equidimensional.

From $\dim(S(S_{u,v})) = 5$ and our assumption that $N \geq 5$, we are able to choose a $(D-N)$-plane which intersect $S(S_{u,v})$ in any one and exactly one point. Therefore, we can find P so that $P^{\perp}$ intersects $S(S_{u,v})$ in one point outside $T(S_{u,v}) \cup Z_2$. This provides an example of (4.8) for $m = r = 1$. It follows that p_2 has degree one, and the image is rational since $\sigma(l_1, l_1')$ is rational by Corollary 4.7. $\square$

4.5. **Proof of Theorem 4.2.**

Lemma 4.10. *Assume $D \geq N \geq 5$, and assume the existence of a degree D rational scroll $S \subset \mathbb{P}^N$ with isolated singularities which has at least r singularities. Then $\mathcal{H}^r_{u,v}$ has codimension at most $r(N-4)$ in $\mathcal{H}_{u,v}$. For $r = 1$, $\mathcal{H}^1_{u,v}$ is unirational of codimension exactly $N-4$.*

Proof. We have the following diagram

$$\begin{tikzcd} & \mathbb{V}(N, D+1) \arrow[r, dashed, "\pi"] \arrow[d, "p"] & \mathcal{H}_{u,v} \\ \Sigma_r \arrow[r, "p_2"] & \mathbb{G}(N, D+1). & \end{tikzcd}$$

By definition, $\mathcal{H}^r_{u,v} = \pi\left(p^{-1}\left(p_2(\Sigma_r)\right)\right)$.

By Lemma 4.9, $p_2(\Sigma_1)$ is rational, which implies that $\mathcal{H}^1_{u,v}$ is unirational.

It's clear that $p_2(\Sigma_r)$ and $p^{-1}(p_2(\Sigma_r))$ have the same codimension. On the other hand, $p^{-1}(p_2(\Sigma_r))$ and $\pi^{-1}\left(\pi\left(p^{-1}\left(p_2(\Sigma_r)\right)\right)\right)$ have the same dimension since both contain an open dense subset consisting of the projections which generate r singularities, so the codimension of $p^{-1}(p_2(\Sigma_r))$ is the same as its image through π. Therefore, $p_2(\Sigma_r)$ and $\mathcal{H}^r_{u,v}$ have the same codimension in their own ambient spaces, and the results follows from Lemma 4.9. □

Lemma 4.10 is the special case of Theorem 4.2 when restricting to the locus of a particular type on the Hilbert scheme. The next lemma shows that a general $S \in \mathcal{H}^r_D$ deforms equisingularly between different types under the assumption $rN \leq (D+2)^2 - 1$. Hence the dimension estimate made by Lemma 4.10 can be extended regardless of the types.

Lemma 4.11. *Assume (4.8) and $rN \leq (D+2)^2 - 1$, then $\mathcal{H}^r_{u,v} = \mathcal{H}_{u,v} \cap \mathcal{H}^r_{u+k,v-k}$ for $0 \leq 2k < v - u$.*

Proof. It is trivial that $\mathcal{H}^r_{u,v} \supset \mathcal{H}_{u,v} \cap \mathcal{H}^r_{u+k,v-k}$. To prove that $\mathcal{H}^r_{u,v} \subset \mathcal{H}_{u,v} \cap \mathcal{H}^r_{u+k,v-k}$, it is sufficient to show that a generic element in $\mathcal{H}^r_{u,v}$ deforms equisingularly to an element in $\mathcal{H}_{u+k,v-k}$.

If $(u,v) = \left(\lfloor \frac{D}{2} \rfloor, \lceil \frac{D}{2} \rceil\right)$ then there is nothing to prove, so we assume $(u,v) \neq \left(\lfloor \frac{D}{2} \rfloor, \lceil \frac{D}{2} \rceil\right)$. The elements satisfying (4.8) form an open dense subset of $\mathcal{H}^r_{u,v}$. Let $S \in \mathcal{H}^r_{u,v}$ be one of them, and assume S is the image of $F \cong S_{u,v} \subset \mathbb{P}^{D+1}$ projected from some $(D-N)$-plane Q. By hypothesis, F has r secants γ_1, ..., γ_r incident to Q. Assume $\gamma_j \cap F = \{x_j, y_j\}$ for $j = 1, ..., r$.

$H^1\left(T_{\mathbb{P}^{D+1}}|_F\right) = 0$ by Lemma 1.3, so the short exact sequence $0 \to T_F \to T_{\mathbb{P}^{D+1}}|_F \to N_{F/\mathbb{P}^{D+1}} \to 0$ induces the exact sequence

$$0 \to H^0(T_F) \to H^0\left(T_{\mathbb{P}^{D+1}}|_F\right) \to H^0\left(N_{F/\mathbb{P}^{D+1}}\right) \to H^1(T_F) \to 0.$$

By Lemma 1.2, $h^1(F, T_F) = h^1\left(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(u-v)\right) = v - u - 1$, the same as the codimension of $\mathcal{H}_{u,v}$ in $\mathcal{H}_D$, thus a deformation normal to $\mathcal{H}_{u,v}$ is induced from an element in $H^1(T_F)$. In order to prove that the deformation is equisingular, it is sufficient to prove that for all $\mathcal{F} \in H^1(T_F)$ and its lift $\mathcal{S} \in H^0\left(N_{F/\mathbb{P}^{D+1}}\right)$, there exists $\alpha \in H^0(T_{\mathbb{P}^{D+1}}|_F)$ such that the vectors $\mathcal{S}(x_j) + \alpha(x_j) \in T_{\mathbb{P}^{D+1},x_j}$ and $\mathcal{S}(y_j) + \alpha(y_j) \in T_{\mathbb{P}^{D+1},y_j}$ keep γ_j contact with Q for $j = 1, ..., r$, so that $\mathcal{S} + \alpha$ is a lift of $\mathcal{F}$ representing an embedded deformation which preserves the incidence of the r secants to Q.

Note that for arbitrary $p \in \mathbb{P}^{D+1}$, the tangent space $T_{\mathbb{P}^{D+1},p} \cong \mathrm{Hom}\,(p, p^\perp) \cong p^\perp$ can be considered as a subspace of $\mathbb{P}^{D+1}$. We identify a point in $\mathbb{P}^{D+1}$ with its underlying vector. Let $\gamma = \gamma_j$ for some j, and let $\{x, y\} = \gamma \cap S_{u,v}$ with $x = (x_1, ..., x_{D+1})$ and $y = (y_1, ..., y_{D+1})$. The condition that $\mathcal{S}(x) + \alpha(x)$ and $\mathcal{S}(y) + \alpha(y)$ keep γ contact with Q is equivalent to the condition that the set of vectors consisting of $x + \mathcal{S}(x) + \alpha(x)$, $y + \mathcal{S}(y) + \alpha(y)$ and the basis of Q is not independent.

One can compute that $h^0(T_{\mathbb{P}^{D+1}}|_F) = (D + 2)^2 - 1$ by the Euler exact sequence $0 \to \mathcal{O}_F \to \mathcal{O}_F(1)^{\oplus(D+2)} \to T_{\mathbb{P}^{D+1}}|_F \to 0$ and Lemma 1.1. Suppose $H^0(T_{\mathbb{P}^{D+1}}|_F)$ has basis e_1, ..., $e_{(D+2)^2-1}$, we write the evaluation of e_i at p as $e_i(p) = (e_i(p)_1, ..., e_i(p)_{D+1})$. Let $\alpha = \sum_{i\geq 1} \alpha_i e_i$, $\mathcal{S}(x) = \sum_{i\geq 1} c_i e_i(x)$ and $\mathcal{S}(y) = \sum_{i\geq 1} d_i e_i(y)$, also write $Q = \left(q_{i,j}\right)$ as a $(D - N + 1) \times (D + 2)$-matrix. Then the dependence condition is equivalent to the condition that the $(D - N + 3) \times (D + 2)$-matrix

$$A_\gamma = \begin{pmatrix} \alpha_0 x + \alpha_0 \mathcal{S}(x) + \alpha(x) \\ \alpha_0 y + \alpha_0 \mathcal{S}(y) + \alpha(y) \\ Q \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0 x_0 + \sum(\alpha_0 c_i + \alpha_i)\, e_i(x)_0 & \dots & \alpha_0 x_{D+1} + \sum(\alpha_0 c_i + \alpha_i)\, e_i(x)_{D+1} \\ \alpha_0 y_0 + \sum(\alpha_0 d_i + \alpha_i)\, e_i(y)_0 & \dots & \alpha_0 y_{D+1} + \sum(\alpha_0 d_i + \alpha_i)\, e_i(y)_{D+1} \\ q_{0,0} & \cdots & q_{0,D+1} \\ \vdots & & \vdots \\ q_{D-N,0} & \cdots & q_{D-N,D+1} \end{pmatrix}$$

has rank at most $D - N + 2$. Here we homogenize the first two rows by α_0, so that the matrix defines a morphism

$$\begin{array}{rccc} \mathbf{A}_\gamma : & \mathbb{P}\left(\mathbb{C} \oplus H^0(T_{\mathbb{P}^{D+1}}|_F)\right) \cong \mathbb{P}^{(D+2)^2-1} & \to & \mathbb{P}^{(D-N+3)(D+2)-1} \\ & (\alpha_0, ..., \alpha_{(D+2)^2-1}) & \mapsto & A_\gamma. \end{array}$$

Let $M_{D-N+2} \subset \mathbb{P}^{(D-N+3)(D+2)-1}$ be the determinantal variety of matrices of rank at most $D - N + 2$. Then $\mathbf{A}_\gamma{}^{-1}(M_{D-N+2}) \subset \mathbb{P}\left(\mathbb{C} \oplus H^0(T_{\mathbb{P}^{D+1}}|_F)\right)$ is an irreducible and nondegenerate subvariety of codimension N, whose locus outside $\alpha_0 = 0$ parametrizes those $\alpha \in H^0(T_{\mathbb{P}^{D+1}}|_F)$ such that $\mathcal{S} + \alpha$ preserves the incidence between γ and Q.

It follows that the intersection $\bigcap_{j=1}^r \mathbf{A}_{\gamma_j}{}^{-1}(M_{D-N+2})$ is nonempty by the hypothesis $rN \leq (D+2)^2 - 1$. Moreover, it is not contained in the hyperplane $\alpha_0 = 0$ for a generic $S \in \mathcal{H}_{u,v}^r$. Indeed, if this doesn't hold, then the limit case $\gamma_1 = ... = \gamma_r$ should also be inside the hyperplane $\alpha_0 = 0$. However, the intersection in that case is a multiple of a nondegenerate variety, a contradiction. As a result, for a generic $S \in \mathcal{H}_{u,v}^r$ we can find α from $\bigcap_{j=1}^r \mathbf{A}_{\gamma_j}{}^{-1}(M_{D-N+2})$ which lies on $\{\alpha_0 = 1\} = H^0(T_{\mathbb{P}^{D+1}}|_F)$, so that $\mathcal{S} + \alpha$ preserves the incidence condition between $\gamma_1, ..., \gamma_r$ and Q. □

Now we are ready to finish the proof of Theorem 4.2.

Note that $\mathcal{H}_D^r = \bigcup_{u+v=D} \mathcal{H}_{u,v}^r$. By Lemma 4.11

$$\bigcup_{u+v=D} \mathcal{H}_{u,v}^r = \bigcup_{u+v=D} \left(\mathcal{H}_{u,v} \cap \mathcal{H}^r_{\lfloor \frac{D}{2} \rfloor, \lceil \frac{D}{2} \rceil} \right) = \mathcal{H}_D \cap \mathcal{H}^r_{\lfloor \frac{D}{2} \rfloor, \lceil \frac{D}{2} \rceil} = \mathcal{H}^r_{\lfloor \frac{D}{2} \rfloor, \lceil \frac{D}{2} \rceil}.$$

Therefore $\mathcal{H}_D^r = \mathcal{H}^r_{\lfloor \frac{D}{2} \rfloor, \lceil \frac{D}{2} \rceil}$, and the result follows from Lemma 4.10 with $(u, v) = (\lfloor \frac{D}{2} \rfloor, \lceil \frac{D}{2} \rceil)$.

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