TRANSVERSE LINEAR SUBSPACES TO HYPERSURFACES OVER FINITE FIELDS

SHAMIL ASGARLI, LIAN DUAN, AND KUAN-WEN LAI

ABSTRACT. Ballico proved that a smooth projective variety X of degree d and dimension m over a finite field of q elements admits a smooth hyperplane section if $q \geq d(d-1)^m$. In this paper, we refine this criterion for higher codimensional linear sections on smooth hypersurfaces and for hyperplane sections on Frobenius classical hypersurfaces. We also prove a similar result for the existence of reduced hyperplane sections on reduced hypersurfaces.

1. INTRODUCTION

A classical theorem of Bertini asserts that a smooth projective variety $X \subset \mathbb{P}^n$ defined over an infinite field k admits a smooth hyperplane section defined over k. By applying this theorem repeatedly, one can obtain a linear section on X of any dimension without extending the ground field k.

If $k = \mathbb{F}_q$ is a finite field, then Bertini's theorem is no longer true in its original form as there are only finitely many hyperplanes in $\mathbb{P}^n$ defined over $\mathbb{F}_q$, and they could all happen to be tangent to X. As a concrete counterexample, see [Kat99, Question 10], [Poo04, Theorem 3.1] or [Asg19a, Example 2.2]. There are two approaches to remedy this situation:

(1) Instead of intersecting X with hyperplanes, one could allow intersection with hypersurfaces of arbitrary degrees. This approach was taken by Poonen in [Poo04], where he proved the existence of a hypersurface Y over the ground field such that the intersection $X \cap Y$ is smooth.

(2) Bertini's theorem is still valid if the cardinality of $\mathbb{F}_q$ is sufficiently large with respect to $d := \deg(X)$. In this direction, Ballico [Bal03] proved that if

$$q \geq d(d-1)^{\dim X},$$

then there exists an hyperplane H over $\mathbb{F}_q$ such that $X \cap H$ is smooth. Applying this result repeatedly, one can obtain a smooth linear section on X over $\mathbb{F}_q$ of any dimension.

In the direction of (2), Cafure–Matera–Privitelli [CMP15, Corollary 6.6] and Matera–Pérez–Privitelli [MPP16, Theorem 3.6] extended Ballico's result to higher codimensional linear sections on possibly singular complete intersections. In the case of smooth hypersurfaces $X \subset \mathbb{P}^n$ of degree d over $\mathbb{F}_q$, both results assert the existence of an r-dimensional linear subspace $L \subset \mathbb{P}^n$ over $\mathbb{F}_q$ such that $X \cap L$ is smooth provided that

$$q \geq d(d-1)^r + f_r(d) \tag{1.1}$$

where $f_r(d)$ is a polynomial in d of degree r with coefficients depending on r.

In this paper, we first establish a statement similar to [CMP15, MPP16] for smooth hypersurfaces via an independent approach. This new method allows us to construct inductively a flag of linear subspaces that satisfy a stronger notion of transversality. Moreover, our lower

2020 Mathematics Subject Classification 14G15, 14J70, 14N05.
Keywords: Bertini's theorem, hypersurfaces, finite fields, Frobenius classicality

bound for q is the same as (1.1) except that $f_r(d)$ is replaced by a constant $\leq d$. In the following, we call an r-dimensional linear subspace in $\mathbb{P}^n$ briefly as an *r-plane*.

Theorem 1.1. *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d defined over $\mathbb{F}_q$ and pick any $0 \leq r \leq n-1$. Suppose that*

$$q \geq d(d-1)^r + \beta_r \quad \textit{where} \quad \beta_r = \begin{cases} 1 & \textit{if} \quad r \leq n-3, \\ d & \textit{if} \quad r = n-2, \\ 0 & \textit{if} \quad r = n-1. \end{cases}$$

*Then there exists a sequence of linear subspaces $H_0 \subset H_1 \subset \cdots \subset H_r$ where each H_i is an i-plane over $\mathbb{F}_q$ that is **very transverse** to X in the following sense:*

- *H_i is transverse to X, that is, $X \cap H_i$ is smooth, and*
- *H_i is contained in a hyperplane over $\overline{\mathbb{F}_q}$ that is transverse to X.*

A transverse hyperplane is automatically very transverse, so Theorem 1.1 recovers Ballico's result when $r = n-1$. For higher codimensions, the notion of very transversality becomes different from the usual transversality. As a simple example, consider the conic $C \subset \mathbb{P}^2$ over a field of characteristic 2 defined by the equation

$$x^2 = yz. \tag{1.2}$$

This is an example of a *strange curve* as the point $[x : y : z] = [1 : 0 : 0]$ lies on every tangent line of C. Notice that this point is not on C, so it represents a transverse 0-plane which is not very transverse. More discussions on very transversality and the proof of Theorem 1.1 will be given in Section 2.

Remark 1.2. For $d \geq 3$ and $r \geq 1$, we can improve $\beta_r = 1$ to $\beta_r = 0$ for $r \leq n-3$. Indeed, the quantity $d(d-1)^r$ is never a prime power in this case, and hence the hypothesis $q \geq d(d-1)^r + 1$ can be relaxed to $q \geq d(d-1)^r$.

Remark 1.3. When $n = 3$ and $r = 1$, we have $\beta_r = d$ and Theorem 1.1 implies that a smooth surface in $\mathbb{P}^3$ admits a transverse $\mathbb{F}_q$-line if $q \geq d(d-1) + d = d^2$. This result was proved using a different idea in our previous paper [ADL21, Theorem 3.1].

The second part of our paper focuses on a special type of hypersurfaces. Given a smooth hypersurface $X \subset \mathbb{P}^n$ over $\mathbb{F}_q$, we call X *Frobenius classical* if there exists a point $P \in X$ whose image under the q-th Frobenius endomorphism is outside the tangent hyperplane $T_P X$. Otherwise, we call X *Frobenius nonclassical*. Note that a hyperplane is Frobenius nonclassical by definition. Examples of Frobenius classical hypersurfaces include reflexive hypersurfaces [ADL21, Theorem 4.5]. We expect such a hypersurface to have a transverse r-plane over $\mathbb{F}_q$ provided that $q \geq \mathrm{O}(d^r)$. As evidences, it is known that

- a Frobenius classical curve $C \subset \mathbb{P}^2$ of degree d over $\mathbb{F}_q$ admits a transverse $\mathbb{F}_q$-line if $q \geq d-1$ [Asg19b, Theorem 3.3.1].
- a Frobenius classical surface $S \subset \mathbb{P}^3$ of degree d over $\mathbb{F}_q$ admits a transverse $\mathbb{F}_q$-line when $q \geq cd$ for some constant $c > 0$ [ADL21, Theorem 0.1].

In Section 3, we prove this conjecture for hyperplane sections on hypersurfaces:

Theorem 1.4. *Let $X \subset \mathbb{P}^n$ be a smooth Frobenius classical hypersurface of degree d defined over $\mathbb{F}_q$. Suppose*

$$q \geq c_d \cdot d(d-1)^{n-2} \qquad \text{where} \qquad c_d = \begin{cases} 1 & \text{for} \quad d = 2 \\ (3d+3)(3d-1)^{-1} & \text{for} \quad d \geq 3. \end{cases}$$

Then there exists an $\mathbb{F}_q$-hyperplane $H \subset \mathbb{P}^n$ such that $X \cap H$ is smooth.

Note that c_d strictly decreases in d for $d \geq 3$ starting from $c_3 = 3/2$, and $c_d \to 1$ as $d \to \infty$. In particular, the statement of Theorem 1.4 still holds if c_d is replaced by the constant $3/2$. Given a fixed ambient dimension n, this theorem improves the bound provided by Ballico's theorem by a factor of $d-1$.

In general, a Bertini type theorem concerns the existence of linear sections that inherit some nice properties such as smoothness, reducedness, irreducibility, and normality, from the ambient variety. Recall that a scheme X is *reduced* if $\mathcal{O}_X(U)$ contains no nonzero nilpotent element for every open subset $U \subset X$ [Har77, II, Section 3]. Because $\mathbb{F}_q$ is perfect, reducedness over the ground field $\mathbb{F}_q$ and the algebraic closure $\overline{\mathbb{F}_q}$ are equivalent. In particular, a hypersurface $X \subset \mathbb{P}^n$ defined by a polynomial $F \in \mathbb{F}_q[x_0, \ldots, x_n]$ is reduced provided that the quotient ring $\overline{\mathbb{F}_q}[x_0, ..., x_n]/(F)$ contains no nonzero nilpotent element. Combining these ingredients, we obtain the criterion: *a hypersurface $X = \{F = 0\} \subset \mathbb{P}^n$ over $\mathbb{F}_q$ is reduced if and only if in the factorization of F into irreducible polynomials over $\overline{\mathbb{F}_q}$*

$$F = \prod_{i=1}^{m} G_i \ \in \ \overline{\mathbb{F}_q}[x_0, \ldots, x_n]$$

all the factors G_i are coprime to each other.

Our third result concerns the existence of reduced hyperplane sections, and can be viewed as Bertini's theorem for reducedness over finite fields.

Theorem 1.5. *Let $X \subset \mathbb{P}^n$ be a reduced hypersurface of degree d over $\mathbb{F}_q$. Then there exists a hyperplane $H \subset \mathbb{P}^n$ over $\mathbb{F}_q$ such that $X \cap H$ is reduced and has dimension $n-2$ if q is greater than or equal to a constant depending only on n and d as given below:*

	$n = 2$	$n = 3$	$n \geq 4$
$q \geq$	$\frac{3}{2}d(d-1)$	$d(d-1)+1$	d

By applying this theorem repeatedly, one can deduce similar statements for higher codimensional linear sections; see Corollary 4.8. Notice that, in the case $n = 2$, the theorem asserts the existence of an $\mathbb{F}_q$-line H in $\mathbb{P}^2$ meeting a reduced plane curve in its smooth locus transversely. Theorem 1.5 will be proved in Section 4.

Acknowledgements. We are very grateful to an anonymous referee for a detailed review and helpful suggestions that improved both the accuracy and clarity of our work. The first author was partially supported by NSF grant 1701659, and by a postdoctoral research fellowship from the University of British Columbia. The third author was supported by the ERC Synergy Grant HyperK (ID: 854361).

2. Existence of very transverse linear subspaces

Assume that one would like to prove Theorem 1.1 for merely transverse linear subspaces by induction on r. Then, given an $(r-1)$-plane $H_{r-1} \subset \mathbb{P}^n$ over $\mathbb{F}_q$ transverse to the smooth hypersurface X, one needs to find an r-plane $H_r \supset H_{r-1}$ also transverse to X. However, such an H_r may not exist in general in view of the strange conic (1.2). In order to remedy this situation, we run the induction process for transverse linear subspaces that satisfy additional properties:

Definition 2.1. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface over an arbitrary field k. We say a linear subspace $H \subset \mathbb{P}^n$ is *very transverse* to X if

(1) it is transverse to X, and
(2) it is contained in a hyperplane over $\overline{k}$ that is transverse to X.

Property (2) can be translated into the following equivalent form in terms of projective duality. Consider the Gauss map

$$\gamma\colon X \longrightarrow (\mathbb{P}^n)^* : P \longmapsto T_P X.$$

Let $X^* := \gamma(X)$ be the projective dual of X and $H^* \subset (\mathbb{P}^n)^*$ be the subspace consisting of the hyperplanes in $\mathbb{P}^n$ that contain H. Then (2) is equivalent to

$$H^* \not\subset X^*.$$

In characteristic zero, a transverse linear subspace is automatically very transverse due to the general version of Bertini's theorem (see [Kle74, Corollary 5]). On the other hand, there exist odd dimensional smooth quadrics in characteristic 2 which admit linear subspaces that are transverse but not very transverse:

Example 2.2. Suppose that $X \subset \mathbb{P}^n$ is a smooth hypersurface of degree at least 2 over an algebraically closed field. Then X is *strange*, meaning that its tangent hyperplanes contain a common point $P \in \mathbb{P}^n$, if and only if it is an odd dimensional quadric in characteristic 2 [KP91, Theorem 7]. In this case, one can express X as

$$x_0^2 + \sum_{i=1}^{m} x_{2i-1}x_{2i} = 0 \quad \text{where} \quad m = \frac{n}{2} \tag{2.1}$$

and let P be the point $[x_0 : x_1 : \cdots : x_n] = [1 : 0 : \cdots : 0]$. Since P is not on X and every hyperplane containing P is tangent to X, it represents a 0-plane that is transverse but not very transverse to X. More generally, for every even $0 \leq r < n$, the r-plane

$$H := \{x_{r+1} = \cdots = x_n = 0\} \cong \mathbb{P}^r$$

is transverse as it intersects X in a strange quadric defined by (2.1) with m replaced by r. Moreover, the hyperplanes containing H also contain P, hence they are tangent to X; consequently, H is not very transverse.

Let us first show that, in the cases $d = 1, 2$, Theorem 1.1 follows easily from the main result of [Bal03].

Proposition 2.3. *Let $X \subset \mathbb{P}^n$ be a hyperplane or a smooth quadric over $\mathbb{F}_q$. Then there exists a sequence of linear subspaces $H_0 \subset H_1 \subset \cdots \subset H_{n-1}$ where each H_r is an r-plane defined over $\mathbb{F}_q$ and very transverse to X.*

Proof. Ballico [Bal03] proved that if $q \geq d(d-1)^{\dim X}$ then there is an $\mathbb{F}_q$-hyperplane H_{n-1} transverse to $X_{n-1} := X$. Note that $d(d-1)^{\dim X}$ equals 0 when $d = 1$ and equals 2 when $d = 2$, so the above inequality always holds in our situation. Therefore, we can take $X_{n-2} := X_{n-1} \cap H_{n-1}$, consider it as a hypersurface in $H_{n-1} \cong \mathbb{P}^{n-1}$, and repeat the same process to find an $(n-2)$-plane $H_{n-2} \subset H_{n-1}$ over $\mathbb{F}_q$ transverse to X_{n-2}. Thus, by induction, we have a sequence of transverse linear subspaces $H_0 \subset H_1 \subset \cdots \subset H_{n-1}$ such that $\dim H_r = r$. Notice that each H_r in this sequence is very transverse since they are all contained in the transverse hyperplane H_{n-1}. □

2.1. **Strategy for proving Theorem 1.1.** Let $X \subset \mathbb{P}^n$ be a smooth hypersurface over $\mathbb{F}_q$ that admits a very transverse $(r-1)$-plane $H_{r-1} \subset \mathbb{P}^n$. Among the r-planes over $\mathbb{F}_q$ that contain H_{r-1}, we would like to estimate the number of bad choices, namely, the r-planes that are *not* very transverse to X.

By definition, a linear subspace $H \subset \mathbb{P}^n$ is not very transverse to X if and only if

(i) it is not transverse to X, or
(ii) the dual subspace $H^* \subset (\mathbb{P}^n)^*$ is contained in X^*.

Our estimates for the numbers of bad r-planes of these two types are established using geometry of the dual hypersurface X^*. Let us deal with r-planes of type (ii) first:

Proposition 2.4. *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d over a field k which admits a very transverse $(r-1)$-plane H_{r-1} over k. Then the number of r-planes H that contain H_{r-1} and satisfy $H^* \subset X^*$ is at most $d(d-1)^{n-1}$.*

Proof. By hypothesis, $H_{r-1}^* \cap X^*$ is a hypersurface in $H_{r-1}^* \cong \mathbb{P}^{n-r}$ whose degree is equal to $\deg(X^*) \leq d(d-1)^{n-1}$ by Plücker's formula [Kle86, Propositions 2 and 9]. For each r-plane H containing H_{r-1}, its dual H^* appears as a hyperplane in H_{r-1}^*, and $H^* \subset X^*$ means that H^* appears as a linear component of the hypersurface $H_{r-1}^* \cap X^*$. As this hypersurface has degree at most $d(d-1)^{n-1}$, the statement follows. □

The estimate for the number of bad r-planes of type (i) is more complicated. We will turn this into a point counting problem on a certain projective scheme, and leave the explicit computation to the next subsection.

In the following, we fix a smooth hypersurface $X \subset \mathbb{P}^n$ of degree d over a field k and assume that it admits a very transverse $(r-1)$-plane H_{r-1} over k. Let $\gamma\colon X \longrightarrow (\mathbb{P}^n)^*$ be the Gauss map associated with X and define

$$Y_{r-1} := \gamma^{-1}(H_{r-1}^*). \tag{2.2}$$

Proposition 2.5. *The scheme Y_{r-1} defined above, as a subscheme of $\mathbb{P}^n$, is a complete intersection of codimension $r+1$ and multidegree $(d, d-1, \ldots, d-1)$.*

Proof. By hypothesis, the intersection $X^* \cap H_{r-1}^*$ is a hypersurface in $H_{r-1}^* \cong \mathbb{P}^{n-r}$, so it has dimension $n-r-1$. Since $Y_{r-1} = \gamma^{-1}(X^* \cap H_{r-1}^*)$, the finiteness of Gauss map [Zak93, Corollary I.2.8] implies that Y_{r-1} has dimension $n-r-1$ as well. Thus it has codimension $r+1$ in $\mathbb{P}^n$. Let $L_i = 0$, $i = 1, \ldots, r$, be the linear equations that cut out H_{r-1}^* in $(\mathbb{P}^n)^*$. By construction, Y_{r-1} is the zero locus on X of the polynomials $\gamma^* L_i$ for $i = 1, \ldots, r$. This shows that Y_{r-1} is a complete intersection, and its multidegree is determined by $\deg(X) = d$ and $\deg(\gamma^* L_i) = d-1$ for all i. □

Let $\pi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r}$ be the projection from H_{r-1}. Note that taking a point $P \in \mathbb{P}^{n-r}$ to its preimage $\pi^{-1}(P) \subset \mathbb{P}^n$ defines a bijection between the following sets

$$\mathbb{P}^{n-r}(k') \xrightarrow{\sim} \{r\text{-planes } H \subset \mathbb{P}^n \text{ over } k' \text{ such that } H \supset H_{r-1}\} \tag{2.3}$$

where k' is any field extension over k. In the following, the *proper image* $\pi(Y_{r-1})$ means the closure of $\pi(Y_r^\circ)$, where $Y_{r-1}^\circ \subset Y_{r-1}$ be the subset where π is well-defined.

Lemma 2.6. *Let $P \in \mathbb{P}^{n-r}$ be a k-point outside the proper image $\pi(Y_{r-1}) \subset \mathbb{P}^{n-r}$ of the scheme defined in (2.2). Then the image r-plane H of P under bijection (2.3) is defined over k and transverse to X.*

Proof. Assume, to the contrary, that H is not transverse to X, namely, there exists a point $Q \in X \cap \pi^{-1}(P)$ such that $Q \in H \subset T_Q X$. The inclusion relation $H_{r-1} \subset H$ implies $H_{r-1} \subset T_Q X$, or equivalently, $\gamma(Q) \in H_{r-1}^*$, which implies $Q \in Y_{r-1}$. We claim that $Q \notin H_{r-1}$. Indeed, if $Q \in H_{r-1}$, then the same relation $H_{r-1} \subset H$ implies $Q \in H_{r-1} \subset T_Q X$, whence H_{r-1} is not transverse to X, contradiction. We conclude that $Q \in Y_{r-1} \setminus H_{r-1}$, so the projection π is well defined at Q, and $P = \pi(Q) \in \pi(Y_{r-1})$, contradiction. □

Suppose that the ground field k is finite. As a consequence of Lemma 2.6, to find an r-plane $H \supset H_{r-1}$ over k which is transverse to X, it is sufficient to show that the k-points on $\mathbb{P}^{n-r}$ are strictly more than the k-points on $\pi(Y_{r-1})$.

2.2. **Proof of Theorem 1.1.** In the following, we will compute an estimate for the number of bad r-planes of type (i) and then conclude the proof of Theorem 1.1. Some technical lemmas needed in the process will be postponed to the next subsection.

Definition 2.7. For every integer $m \geq 0$ and q a power of a prime number, we define

$$\theta_m(q) := \#\mathbb{P}^m(\mathbb{F}_q) = \frac{q^{m+1}-1}{q-1}$$

and set $\theta_m(q) = 0$ for $m < 0$. We will write $\theta_m(q)$ briefly as θ_m if there is no ambiguity about q from the context.

Lemma 2.8. *Let $X \subset \mathbb{P}^n$ be a reduced subscheme over $\mathbb{F}_q$ and write $X = \bigcup_{i=1}^t X_i$ where X_i is an irreducible component of dimension $m_i < n$ and degree d_i. Denote $d := \sum_{i=1}^t d_i$ and $m := \max(m_1, ..., m_t)$. Then*

$$\#X(\mathbb{F}_q) \leq d(\theta_m - \theta_{2m-n}) + \theta_{2m-n}.$$

Proof. We use the inequality from [Cou16, Theorem 3.1], which states

$$\#X(\mathbb{F}_q) \leq \sum_{i=1}^t d_i(\theta_{m_i} - \theta_{2m_i-n}) + \theta_{2m-n}.$$

Based on this, it is sufficient to verify that

$$\theta_{m_i} - \theta_{2m_i-n} \leq \theta_m - \theta_{2m-n} \qquad \text{for each} \qquad i \in \{1, \ldots, t\} \tag{2.4}$$

Let us proceed by three cases:

Case ($2m - n < 0$): Then $2m_i - n < 0$ and (2.4) reduces to $\theta_{m_i} \leq \theta_m$ which clearly holds.

Case ($2m-n\geq 0$ **and** $2m_i-n\geq 0$)**:** Rearrange (2.4) as $\theta_{2m-n}-\theta_{2m_i-n}\leq \theta_m-\theta_{m_i}$, which is the same as

$$\frac{q^{2m}-q^{2m_i}}{q^n}\leq q^m-q^{m_i}, \qquad \text{or equivalently,} \qquad q^m+q^{m_i}\leq q^n.$$

The last inequality holds since $q^m+q^{m_i}\leq q^{n-1}+q^{n-1}=2q^{n-1}\leq q^n$.

Case ($2m-n\geq 0$ **and** $2m_i-n<0$)**:** Then (2.4) reduces to $\theta_{m_i}+\theta_{2m-n}\leq\theta_m$, that is,

$$q^{m_i+1}+q^{2m-n+1}\leq q^{m+1}+1.$$

The hypothesis implies $2m-n>2m_i-n$ and thus $m\geq m_i+1$. We also have $n\geq m+1$, hence $m\geq 2m-n+1$. Therefore,

$$q^{m+1}+1\geq q^m+q^m+1\geq q^{m_i+1}+q^{2m-n+1}+1>q^{m_i+1}+q^{2m-n+1}$$

as desired. □

Lemma 2.9. *Let $X\subset\mathbb{P}^n$ be a smooth hypersurface of degree $d\geq 2$ over $\mathbb{F}_q$. Assume that $1\leq r\leq n-1$ and that X admits a very transverse $(r-1)$-plane $H_{r-1}\subset\mathbb{P}^n$ over $\mathbb{F}_q$. Then X admits a very transverse r-plane $H_r\supset H_{r-1}$ over $\mathbb{F}_q$ provided that*

$$q\geq d(d-1)^r+\beta_r \qquad \textit{where} \qquad \beta_r=\begin{cases}1 & \textit{if} \quad r\leq n-3,\\ d & \textit{if} \quad r=n-2,\\ 0 & \textit{if} \quad r=n-1.\end{cases}$$

Proof. Let $\gamma\colon X\longrightarrow(\mathbb{P}^n)^*$ be the Gauss map associated with X. By Proposition 2.5, the preimage $Y_{r-1}:=\gamma^{-1}(H_{r-1}^*)$ is a complete intersection in $\mathbb{P}^n$ of dimension $n-r-1$ and of degree $d(d-1)^r$. Let $\pi\colon\mathbb{P}^n\dashrightarrow\mathbb{P}^{n-r}$ be the projection from H_{r-1} and consider the proper image $\pi(Y_{r-1})\subset\mathbb{P}^{n-r}$. If we write $\pi(Y_{r-1})=\bigcup_{i=1}^t Y_i'$, where Y_i' is an irreducible component of dimension m_i and degree d_i, then

$$m:=\max(m_1,\dots,m_t)\leq n-r-1 \qquad \text{and} \qquad \sum_{i=1}^t d_i\leq d(d-1)^r.$$

It follows from Lemma 2.8 that

$$\#\pi(Y_{r-1})(\mathbb{F}_q)\leq d(d-1)^r\,(\theta_m-\theta_{2m-n})+\theta_{2m-n}.$$

According to Lemma 2.6, the $\mathbb{F}_q$-points in $\mathbb{P}^{n-r}$ outside $\pi(Y_{r-1})$ one-to-one correspond to the r-planes containing H_{r-1} over $\mathbb{F}_q$ that are transverse to X. On the other hand, there are at most $d(d-1)^{n-1}$ many r-planes $H_r\supset H_{r-1}$ that satisfy $H_r^*\subset X^*$ by Proposition 2.4. Also recall that the transverse hyperplanes are automatically very transverse. As a consequence, there exists H_r as in the statement provided that

$$\theta_{n-r}>d(d-1)^r\,(\theta_m-\theta_{2m-n})+\theta_{2m-n}+\delta_r \tag{2.5}$$

where $\delta_r=d(d-1)^{n-1}$ if $r\leq n-2$ and $\delta_r=0$ if $r=n-1$. The proof of (2.5) is purely numerical and will be established via Lemmas 2.10 and 2.11. □

Proof of Theorem 1.1. The cases $d=1,2$ are already proved in Proposition 2.3, so we assume $d\geq 3$ in the following. To prove the theorem, we will establish the existence of linear subspaces $H_0\subset\cdots\subset H_s$, where each H_i is a very transverse i-plane over $\mathbb{F}_q$, for all $0\leq s\leq r$ by induction on s.

In the initial case $s = 0$, we need to find a point $P \in \mathbb{P}^n(\mathbb{F}_q) \setminus X(\mathbb{F}_q)$. By the Homma–Kim bound [HK13, Theorem 1.2],

$$\#X(\mathbb{F}_q) \le (d-1)q^{n-1} + dq^{n-2} + \theta_{n-3}.$$

It is sufficient to prove that

$$(d-1)q^{n-1} + dq^{n-2} + \theta_{n-3} < \theta_n.$$

A straightforward computation reduces the last inequality to $d - 1 < q$, which follows from our hypothesis since

$$q \ge d(d-1)^r + \beta_r \ge d > d-1.$$

Hence there exists a point $P \in \mathbb{P}^n(\mathbb{F}_q) \setminus X(\mathbb{F}_q)$. Note that P is non-strange, namely, is contained in a transverse hyperplane over $\overline{\mathbb{F}_q}$ because of $d \ge 3$. Therefore, P is a 0-dimensional linear subspace very transverse to X.

Before entering the inductive step, let us prove that

$$d(d-1)^\ell + \beta_\ell \;\ge\; d(d-1)^{\ell-1} + \beta_{\ell-1} \quad \text{for all} \quad 1 \le \ell \le r. \tag{2.6}$$

This implication holds for all $\ell \le n-3$ because $\beta_\ell = \beta_{\ell-1} = 1$ in these cases. It also holds for $\ell = n-2$ because $\beta_{n-2} = d > 1 = \beta_{n-3}$. When $\ell = n-1$, the desired inequality

$$d(d-1)^{n-1} \;\ge\; d(d-1)^{n-2} + d$$

can be derived easily from the assumption $d \ge 3$. As a consequence, combining (2.6) with the hypothesis $q \ge d(d-1)^r + \beta_r$ gives us

$$q \ge d(d-1)^s + \beta_s \quad \text{for all} \quad 0 \le s \le r. \tag{2.7}$$

Now assume that there exists a sequence of linear subspaces $H_0 \subset \cdots \subset H_{s-1}$ for some $s \in \{1, \dots, r\}$ where each H_i is a very transverse i-plane over $\mathbb{F}_q$. Then Lemma 2.9, together with (2.7), implies that there exists a very transverse s-plane $H_s \supset H_{s-1}$ over $\mathbb{F}_q$. This establishes the inductive step and thus finishes the proof. □

2.3. **Some numerical lemmas.** Here we establish inequality (2.5) which is needed in the proof of Lemma 2.9.

Lemma 2.10. *Let n, r, d be positive integers that satisfy $1 \le r \le n-1$ and $d \ge 2$. Define*

$$\delta_r := \begin{cases} d(d-1)^{n-1} & \text{if} \quad r \le n-2 \\ 0 & \text{if} \quad r = n-1 \end{cases} \qquad \text{and} \qquad \beta_r := \begin{cases} 1 & \text{if} \quad r \le n-3 \\ d & \text{if} \quad r = n-2 \\ 0 & \text{if} \quad r = n-1 \end{cases}$$

Then, for every integer q, the inequality $q \ge d(d-1)^r + \beta_r$ implies $(q-1)q^{-n+r}\delta_r \le \beta_r$.

Proof. The implication is obvious when $r = n-1$ since $\delta_r = \beta_r = 0$ in this case. Assume $r \le n-2$. Using $q \ge d(d-1)^r + \beta_r$, we obtain

$$\begin{aligned} (q-1)q^{-n+r}\delta_r < q \cdot q^{-n+r}\delta_r = \frac{\delta_r}{q^{n-r-1}} &\le \frac{\delta_r}{(d(d-1)^r + \beta_r)^{n-r-1}} \\ &\le \frac{\delta_r}{(d(d-1)^r)^{n-r-1}} = \frac{d(d-1)^{n-1}}{(d(d-1)^r)^{n-r-1}}. \end{aligned} \tag{2.8}$$

When $r = n-2$, the last term equals $d-1 < d = \beta_r$, so the statement holds. Assume $r \le n-3$. Together with $r \ge 1$, we obtain $n \ge r+3 \ge r+2+1/r$. It follows that

$$n - r - 1 \ge 1 + \frac{1}{r}, \qquad \text{or equivalently,} \qquad r(n-r-1) \ge r+1. \tag{2.9}$$

The last term of (2.8) can be rewritten as

$$\frac{d(d-1)^{n-1}}{d \cdot d^{n-r-2}(d-1)^{r(n-r-1)}} < \frac{d(d-1)^{n-1}}{d \cdot (d-1)^{n-r-2}(d-1)^{r(n-r-1)}} = \frac{(d-1)^{r+1}}{(d-1)^{r(n-r-1)}}.$$

Then (2.9) implies that the last term is at most $1 = \beta_r$. This establishes the statement. □

Lemma 2.11. *Retain the hypothesis from Lemma 2.10 and assume that q is a power of a prime number. Let m be an integer that satisfies $0 \le m \le n-r-1$. Then*

$$\theta_{n-r}(q) > d(d-1)^r \left(\theta_m(q) - \theta_{2m-n}(q)\right) + \theta_{2m-n}(q) + \delta_r.$$

Proof. We proceed by two cases:

Case $(2m-n \ge 0)$: In this case, the desired inequality is

$$\frac{q^{n-r+1}-1}{q-1} > d(d-1)^r \left(\frac{q^{m+1}-1}{q-1} - \frac{q^{2m-n+1}-1}{q-1}\right) + \frac{q^{2m-n+1}-1}{q-1} + \delta_r.$$

Multiplying both sides by $q^{-n+r}(q-1)$ and rearranging, we obtain

$$q > d(d-1)^r \left(q^{m-n+r+1} - q^{2m-2n+r+1}\right) + q^{2m-2n+r+1} + (q-1)q^{-n+r}\delta_r. \tag{2.10}$$

Consider the right hand side as a function $g(m)$ in m. Taking derivative gives

$$g'(m) = d(d-1)^r \left(q^{m-n+r+1}\ln(q) - 2q^{2m-2n+r+1}\ln(q)\right) + 2q^{2m-2n+r+1}\ln(q).$$

The assumption $m \le n-r-1$ implies $n \ge m+r+1 > m+1$. It follows that

$$m - n + r = m + (n-2n) + r > 2m - 2n + r + 1.$$

Hence $q^{m-n+r+1} \ge 2q^{m-n+r} > 2q^{2m-2n+r+1}$, which implies $g'(m) > 0$, so $g(m)$ is increasing in m. As a consequence, it is sufficient to prove (2.10) when m attains the maximal possible value $n-r-1$, that is,

$$q > d(d-1)^r \left(1 - q^{-r-1}\right) + q^{-r-1} + (q-1)q^{-n+r}\delta_r.$$

We establish this via the following inequalities:

$$\begin{aligned} q &\ge d(d-1)^r + \beta_r > d(d-1)^r + q^{-r-1}\underbrace{\left(1 - d(d-1)^r\right)}_{<0} + \beta_r \\ \text{(by Lemma 2.10)} \qquad &\ge d(d-1)^r - q^{-r-1}d(d-1)^r + q^{-r-1} + (q-1)q^{-n+r}\delta_r \\ &= d(d-1)^r \left(1 - q^{-r-1}\right) + q^{-r-1} + (q-1)q^{-n+r}\delta_r. \end{aligned}$$

Case $(2m-n < 0)$: In this case, the desired inequality reduces to:

$$\frac{q^{n-r+1}-1}{q-1} > d(d-1)^r \cdot \left(\frac{q^{m+1}-1}{q-1}\right) + \delta_r.$$

Multiplying both sides by $q^{-n+r}(q-1)$ and rearranging the terms gives

$$q > d(d-1)^r q^{m+1-n+r} + (1 - d(d-1)^r)q^{-n+r} + (q-1)q^{-n+r}\delta_r.$$

Notice that $m \leq n-r-1$ implies $m+1-n+r \leq 0$. Then the above inequality follows from

$$\begin{aligned} q &\geq d(d-1)^r + \beta_r \geq d(d-1)^r q^{m+1-n+r} + \beta_r \\ \text{(by Lemma 2.10)} \qquad &> d(d-1)^r q^{m+1-n+r} + \underbrace{\left(1-d(d-1)^r\right)}_{<0} q^{-n+r} + (q-1)q^{-n+r}\delta_r. \end{aligned}$$

This completes the proof. □

3. Smooth sections on Frobenius classical hypersurfaces

We prove Theorem 1.4 in this section. In order to prove this result, we need to estimate the number of hyperplanes over the ground field which are tangent to a smooth hypersurface X. Our main strategy is to turn counting such hyperplanes into counting points on a certain 0-dimensional subscheme of X.

3.1. **Tangent hyperplanes over the ground field.** Let $X \subset \mathbb{P}^n$ be a hypersurface defined over $\overline{\mathbb{F}_q}$, where q is a fixed prime power, and let $F = F(x_0, \ldots, x_n)$ be the defining polynomial of X. Consider the $2 \times (n+1)$ matrix

$$M := \begin{pmatrix} F_0 & \cdots & F_n \\ F_0^q & \cdots & F_n^q \end{pmatrix} \qquad \text{where} \qquad F_i := \frac{\partial F}{\partial x_i}.$$

For each (i,j) such that $0 \leq i < j \leq n$, the maximal minor given by the i-th and j-th columns of this matrix determines a hypersurface

$$D_{ij} := \{F_i F_j^q - F_i^q F_j = 0\} \subset \mathbb{P}^n$$

of degree $(d-1)(q+1)$ over $\overline{\mathbb{F}_q}$. Let us define

$$Z_X := X \cap \bigcap_{0 \leq i < j \leq n} D_{ij}.$$

Proposition 3.1. *Let $X \subset \mathbb{P}^n$ be a hypersurface over $\overline{\mathbb{F}_q}$. Then a point $P \in X(\overline{\mathbb{F}_q})$ belongs to Z_X if and only if P is a singular point of X or $T_P X$ is defined over $\mathbb{F}_q$.*

Proof. Observe that $P \in Z_X$ if and only if the matrix M has rank equal to 0 or 1 when evaluated at P, which happens if and only if

$$(F_0(P), \ldots, F_n(P)) = (0, \ldots, 0) \quad \text{or} \quad [F_0(P) : \cdots : F_n(P)] \in \mathbb{P}^n(\mathbb{F}_q),$$

and thus correspond to the two conditions in the statement. □

Remark 3.2. Here is a more geometric way to view the locus $Z_X \subset X$. Let $\{y_0, \ldots, y_n\}$ be a system of homogeneous coordinates for $(\mathbb{P}^n)^*$. Then D_{ij} is the pullback of the hypersurface $\{y_i y_j^q - y_i^q y_j = 0\} \subset (\mathbb{P}^n)^*$ under the rational map

$$\widetilde{\gamma} \colon \mathbb{P}^n \dashrightarrow (\mathbb{P}^n)^* : [x_0 : \cdots : x_n] \mapsto [F_0 : \cdots : F_n]$$

which is obtained by extending the Gauss map of X to the ambient $\mathbb{P}^n$. Because the intersection $\cap_{i<j}\{y_i y_j^q - y_i^q y_j = 0\}$ defines the collection of $\mathbb{F}_q$-points on $(\mathbb{P}^n)^*$, the locus $Z_X \subset X$ consists of $P \in X$ such that $T_P X$ is defined over $\mathbb{F}_q$. Note that this includes the case $T_P X = \mathbb{P}^n$, that is, P is singular.

Corollary 3.3. *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface over $\overline{\mathbb{F}_q}$. Then $Z_X \subset X$ is a zero-dimensional subscheme which consists of $P \in X$ such that $T_P X$ is defined over $\mathbb{F}_q$. Furthermore, the number of hyperplanes over $\mathbb{F}_q$ tangent to X is bounded by $\# Z_X(\overline{\mathbb{F}_q})$.*

Proof. Since X is smooth, its Gauss map is finite [Zak93, Corollary I.2.8], which implies that Z_X has dimension zero. By Proposition 3.1 and the smoothness, Z_X consists of $P \in X$ such that $T_P X$ is defined over $\mathbb{F}_q$. There exists a surjective map from $Z_X(\overline{\mathbb{F}_q})$ to the set of $\mathbb{F}_q$-hyperplanes tangent to X which sends P to $T_P X$, which proves the last assertion. □

3.2. **Interplay with Frobenius classicality.** Let $X \subset \mathbb{P}^n$ be a smooth hypersurface over $\mathbb{F}_q$ with defining polynomial $F = F(x_0, \ldots, x_n)$. Consider the hypersurface

$$X_{1,0} := \left\{ \sum_{i=0}^{n} x_i^q \frac{\partial F}{\partial x_i} = 0 \right\} \subset \mathbb{P}^n.$$

If we let $\Phi \colon \mathbb{P}^n \longrightarrow \mathbb{P}^n$ denote q-th Frobenius endomorphism, then

$$(X \cap X_{1,0})(\overline{\mathbb{F}_q}) = \left\{ P \in X(\overline{\mathbb{F}_q}) \;\middle|\; \Phi(P) \in T_P X \right\}.$$

In particular, X is Frobenius classical if and only if $X_{1,0}$ does not contain X, that is, $X \cap X_{1,0}$ has dimension $n-2$. On the other hand, Corollary 3.3 asserts that Z_X consists of $P \in X$ such that $\Phi(T_P X) = T_P X$, whence $Z_X \subset X \cap X_{1,0}$. We will use this relation to bound the number $\#Z_X(\overline{\mathbb{F}_q})$, which will then provide a bound for the number of $\mathbb{F}_q$-hyperplanes tangent to X.

Lemma 3.4. *Let X be a smooth hypersurface over $\mathbb{F}_q$. Suppose that $Y \subset X$ is a subscheme which is irreducible over $\mathbb{F}_q$ and of dimension at least 1. Then there exists D_{ij} such that* $\dim(Y \cap D_{ij}) = \dim(Y) - 1$

Proof. Assume, to the contrary, that $\dim(Y \cap D_{ij}) = \dim(Y)$ for all $i < j$. Because each D_{ij} is defined over $\mathbb{F}_q$, the irreducibility of Y over $\mathbb{F}_q$ implies that $Y \subset D_{ij}$. We conclude that Y is contained in $X \cap \bigcap_{i<j} D_{ij} = Z_X$, which is impossible as $\dim(Y) \geq 1$ and $\dim(Z_X) = 0$. □

Proposition 3.5. *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d over $\mathbb{F}_q$. Suppose that $Y \subset X$ is an equidimensional subscheme over $\mathbb{F}_q$ of dimension at least 1. Then $Y \cap Z_X$ consists of at most* $\deg(Y)[(d-1)(q+1)]^{\dim(Y)}$ *many $\overline{\mathbb{F}_q}$-points counted with multiplicity.*

Proof. Let us proceed by induction on $\dim(Y)$. We first establish the inductive step. Assume that the statement holds for any equidimensional subscheme of X over $\mathbb{F}_q$ of dimension ℓ for all $1 \leq \ell < \dim(Y)$. If Y is irreducible over $\mathbb{F}_q$, then Lemma 3.4 shows that there exists D_{ij} such that $\dim(Y \cap D_{ij}) = \dim(Y) - 1$. By the induction hypothesis, the intersection $(Y \cap D_{ij}) \cap Z_X = Y \cap Z_X$ consists of at most

$$\deg(Y \cap D_{ij})[(d-1)(q+1)]^{\dim(Y)-1} = \deg(Y)[(d-1)(q+1)]^{\dim(Y)}$$

many $\overline{\mathbb{F}_q}$-points counted with multiplicity. If Y is not irreducible over $\mathbb{F}_q$, then we can express $Y = \bigcup_{s=1}^{m} Y_s$ where each Y_s is a component irreducible over $\mathbb{F}_q$ and $m \geq 2$. By applying the above result to each Y_s, we conclude that $Y \cap Z_X$ consists of at most

$$\sum_{s=1}^{m} \deg(Y_s)[(d-1)(q+1)]^{\dim(Y_s)} = \deg(Y)[(d-1)(q+1)]^{\dim(Y)}$$

many $\overline{\mathbb{F}_q}$-points counted with multiplicity.

The argument for the initial case $\dim(Y) = 1$ is almost the same as the inductive step. The only difference is that the induction hypothesis has to be replaced by Bézout's theorem in order to conclude that $Y \cap D_{ij}$, and thus $Y \cap Z_X$, consists of at most

$$\deg(Y)\deg(D_{ij}) = \deg(Y)(d-1)(q+1)$$

many $\overline{\mathbb{F}_q}$-points counted with multiplicity. □

Corollary 3.6. *Let $X \subset \mathbb{P}^n$ be a smooth Frobenius classical hypersurface of degree d over $\mathbb{F}_q$ with $n \geq 3$. Then Z_X consists of at most*

$$d(q+d-1)[(d-1)(q+1)]^{n-2}$$

many $\overline{\mathbb{F}_q}$-points counted with multiplicity. In particular, the number of $\mathbb{F}_q$-hyperplanes tangent to X is bounded by the number above.

Proof. By applying Proposition 3.5 to $Y = X \cap X_{1,0}$ which has dimension at least 1 since $n \geq 3$. Since $Y \cap Z_X = (X \cap X_{1,0}) \cap Z_X = Z_X$, we conclude that Z_X has at most

$$\deg(X)\deg(X_{1,0})[(d-1)(q+1)]^{n-2} = d(q+d-1)[(d-1)(q+1)]^{n-2}$$

many $\overline{\mathbb{F}_q}$-points counted with multiplicity. The last assertion follows from Corollary 3.3. □

3.3. **Refinement of Ballico's result.** Let us finish the proof of Theorem 1.4.

Lemma 3.7. *Let $n \geq 2$, $d \geq 3$ be integers and define $c_d := (3d+3)(3d-1)^{-1}$. Then*

$$c_d \cdot d(d-1)^{n-2} \geq 3d(n-2).$$

Proof. For $d = 3$, we have $c_3 = 3/2$, hence $c_3 \cdot d(d-1)^{n-2} = (3/2) \cdot d \cdot 2^{n-2} \geq 3d(n-2)$ where the last inequality uses $2^{n-2} \geq 2(n-2)$. For $d \geq 4$, we have $c_d > 1$. Using the fact that $3^{n-2} \geq 3(n-2)$, we obtain $c_d \cdot d(d-1)^{n-2} > d \cdot 3^{n-2} \geq 3d(n-2)$. □

Proof of Theorem 1.4. The case $d = 2$ follows from Ballico's theorem (see also Proposition 2.3), so we assume $d \geq 3$ in the remaining part of the proof. When $n = 2$, the conclusion follows from [Asg19b, Theorem 3.3.1] which proved existence of a transverse $\mathbb{F}_q$-line assuming $q \geq d-1$, so we may assume $n \geq 3$. By Corollary 3.6, there exists an $\mathbb{F}_q$-hyperplane transverse to X if

$$q^n + q^{n-1} + \cdots + 1 > d(d+q-1)[(d-1)(q+1)]^{n-2}.$$

It suffices to show that

$$q^n \geq d(d+q-1)[(d-1)(q+1)]^{n-2}.$$

Since $q \geq c_d \cdot d(d-1)^{n-2}$, it is enough to show that

$$q^{n-1} \cdot c_d \geq (d+q-1)(q+1)^{n-2},$$

or equivalently,

$$\left(1 - \frac{1}{q+1}\right)^{n-2} \cdot c_d \geq \frac{q+d-1}{q}. \tag{3.1}$$

Our hypothesis and Lemma 3.7 imply $q \geq 3d(n-2)$, thus $\frac{1}{3d} > \frac{n-2}{q+1}$. Therefore,

$$\frac{1}{1-\frac{1}{3d}} > \frac{1}{1-\frac{n-2}{q+1}}. \tag{3.2}$$

Using Bernoulli's inequality [MP93], which asserts that $(1+x)^\ell \geq 1+\ell x$ for all integer $\ell \geq 0$ and real number $x \geq -1$, we obtain:

$$\left(1 - \frac{1}{q+1}\right)^{n-2} c_d \geq \left(1 - \frac{n-2}{q+1}\right) c_d = \left(1 - \frac{n-2}{q+1}\right) \frac{3d+3}{3d-1}$$

$$= \left(1 - \frac{n-2}{q+1}\right)\left(1 + \frac{1}{d}\right)\left(\frac{1}{1-\frac{1}{3d}}\right)$$

(by (3.2)) $$> \left(1 - \frac{n-2}{q+1}\right)\left(1 + \frac{1}{d}\right)\left(\frac{1}{1-\frac{n-2}{q+1}}\right)$$

$$= 1 + \frac{1}{d} \geq 1 + \frac{1}{d(d-1)^{n-3}}$$

(since $q > d(d-1)^{n-2}$) $$> 1 + \frac{1}{q/(d-1)} = 1 + \frac{d-1}{q} = \frac{q+d-1}{q}.$$

This proves the desired inequality (3.1), and completes the proof of the theorem. □

4. Bertini theorems for reduced hypersurfaces

In this section, we prove Theorem 1.5 by splitting the task into two parts. The first result handles the case $n \geq 3$.

Theorem 4.1. *Let $X \subset \mathbb{P}^n$ be a reduced hypersurface of degree $d \geq 2$ over $\mathbb{F}_q$ where $n \geq 3$. Then there exists a hyperplane $H \subset \mathbb{P}^n$ over $\mathbb{F}_q$ such that $X \cap H$ is reduced and has dimension $n-2$ provided that*

- *$q \geq d(d-1)+1$ when $n = 3$,*
- *$q \geq d$ when $n \geq 4$.*

The next result handles the case $n = 2$.

Theorem 4.2. *Let $C \subset \mathbb{P}^2$ be a reduced curve of degree $d \geq 2$ over $\mathbb{F}_q$. Suppose that*

$$q \geq \frac{3}{2}d(d-1).$$

Then there exists an $\mathbb{F}_q$-line $L \subset \mathbb{P}^2$ which is transverse to C.

4.1. **Existence of reduced hyperplane sections.** The next lemma uses the scheme Z_X defined previously in Section 3.1. Since X is not necessarily smooth, it is possible that $\dim(Z_X) \geq 1$.

Lemma 4.3. *Let $X \subset \mathbb{P}^n$ be a reduced hypersurface and $X' \subset X_{\overline{\mathbb{F}_q}}$ be an irreducible component of degree at least 2. Then $X' \not\subset Z_X$. In particular, there exists D_{ij} such that $X' \cap D_{ij}$ has dimension $n-2$.*

Proof. Assume, to the contrary, that $X' \subset Z_X$. Then, for each $P \in X'$, Proposition 3.1 implies that either P is a singular point or $T_P X$ is defined over $\mathbb{F}_q$. Therefore, X' is contained in the union of the singular locus $\mathrm{Sing}(X)$ and all the hyperplanes over $\mathbb{F}_q$. It follows that $X' \subset \mathrm{Sing}(X)$ because X' is irreducible of degree at least 2. However, X being reduced implies that $\mathrm{Sing}(X)$ has dimension at most $n-2$, which leads to a contradiction as $\dim(X') = n-1$. We conclude that $X' \not\subset Z_X$, and the last statement follows as X' is irreducible. □

Let $X \subset \mathbb{P}^n$ be a reduced hypersurface of degree d over $\mathbb{F}_q$. Write $X_{\overline{\mathbb{F}_q}} = \cup_{i=1}^{\ell} X_i$ where each X_i is irreducible and set $d_i := \deg(X_i)$. After rearranging the indices, we may assume there exists $0 \leq t \leq \ell$ such that $d_i = 1$ for $1 \leq i \leq t$ and $d_i > 1$ otherwise. To find a reduced hyperplane section on X over $\mathbb{F}_q$, we need to estimate the number of *bad* hyperplanes, namely,

the hyperplanes H over $\mathbb{F}_q$ which is a component of X, or intersects X properly with the intersection $X \cap H$ being non-reduced. For such an H, let us consider the following three mutually exclusive conditions:

(I) H is a linear component of X, so that $H = X_i$ for some $1 \leq i \leq t$.

In the next two conditions, we assume that H intersects X properly such that $X \cap H$ is non-reduced. If X is defined by the polynomial F, the properness implies that $X \cap H$ is a hypersurface in $H \cong \mathbb{P}^{n-1}$ defined by $F|_H = 0$. Consider the factorization of $F|_H$ over the algebraic closure:

$$F|_H = \prod_{j=1}^{m} G_j^{m_j} \tag{4.1}$$

where the factors G_j are irreducible over $\overline{\mathbb{F}_q}$ and coprime to each other. As explained before Theorem 1.5 in Section 1, the assumption that $X \cap H$ is non-reduced means that $m_j \geq 2$ for some $1 \leq j \leq m$.

(II) H intersects X properly and passes through the intersection of two distinct linear components $X_i \cap X_j$ where $1 \leq i < j \leq t$. In this case, $(X \cap H)_{\overline{\mathbb{F}_q}}$ has $X_i \cap X_j$ as an irreducible component along which $(X \cap H)_{\overline{\mathbb{F}_q}}$ is non-reduced. More explicitly, this means that there exists a factor G_α with $\deg(G_\alpha) = 1$ and $m_\alpha \geq 2$ in (4.1) such that $\{G_\alpha = 0\} = X_i \cap X_j$ as hyperplanes in $H \cong \mathbb{P}^{n-1}$.

(III) H intersects X properly and does not pass through the intersection of any two linear components of X. Hence, if Y is a component of $(X \cap H)_{\overline{\mathbb{F}_q}}$ along which $(X \cap H)_{\overline{\mathbb{F}_q}}$ is non-reduced, then $Y \subset X_i$ for some $i > t$. Note that $Y = \{G_\alpha = 0\}$ where G_α is a factor in (4.1) with $m_\alpha \geq 2$.

Lemma 4.4. *The number of hyperplanes of type (I) or (II) is bounded by*

$$\binom{t}{2} \cdot (q+1) + 1 = \frac{1}{2} t(t+1)(q+1) + 1.$$

Proof. Suppose that $t \leq 1$, that is, X contains at most one linear component. Then there is at most one hyperplane of type (I) and no hyperplane of type (II). Hence the number of hyperplanes under consideration is bounded by 1.

Suppose that X contains at least two linear components. Then a hyperplane H of type (I) or (II) must contain $X_i \cap X_j$ for some $1 \leq i < j \leq t$. Note that there are at most $q+1$ many $\mathbb{F}_q$-hyperplanes pass through $X_i \cap X_j$ due to, for example, the fact that the intersection of more than $q+1$ many $\mathbb{F}_q$-hyperplanes in $\mathbb{P}^n$ has codimension at least 3. Therefore, the bound in this case is given by $\binom{t}{2} \cdot (q+1)$.

Adding the two bounds in the above two cases together gives the desired bound. □

Given a hyperplane H of type (III), we define $\mathcal{A}_H$ to be the collection of subschemes $Y \subset X$ which can be written as $Y = \{G_\alpha = 0\}$ for some G_α with $m_\alpha \geq 2$ in (4.1). Given any $Y \in \mathcal{A}_H$ and any $P \in Y(\overline{\mathbb{F}_q})$, either $T_P X = H$ or P is a singular point of X. It then follows from Proposition 3.1 that

$$Y \subset Z_X = X \cap \bigcap_{0 \leq i < j \leq n} D_{ij}. \tag{4.2}$$

Let us further take the union

$$\mathcal{B} := \bigcup_{H \text{ satisfies (III)}} \mathcal{A}_H.$$

Lemma 4.5. *The cardinality of $\mathcal{B}$ is bounded by $(d-t)(d-1)(q+1)$. Therefore, the number of hyperplanes of type (III) is bounded by $(d-t)(d-1)(q+1)^2$.*

Proof. Let $Y \in \mathcal{B}$. Then $Y \subset X_k$ for some $k \geq t+1$. Because $\deg(X_k) \geq 2$, there exists D_{ij} such that $\dim(X_k \cap D_{ij}) = n-2$ by Lemma 4.3. Hence Y is the underlying reduced subscheme of an irreducible component of $(X_k \cap D_{ij})_{\overline{\mathbb{F}_q}}$ due to (4.2). By Bézout's theorem, the number of irreducible components of $(X_k \cap D_{ij})_{\overline{\mathbb{F}_q}}$ is bounded by

$$\deg(X_k \cap D_{ij}) = \deg(X_k)\deg(D_{ij}) = d_k(d-1)(q+1).$$

Hence the cardinality of $\mathcal{B}$ is bounded by

$$\sum_{k=t+1}^{\ell} d_k(d-1)(q+1) = (d-t)(d-1)(q+1).$$

This proves the first statement. The second statement holds since there are at most $q+1$ many $\mathbb{F}_q$-hyperplanes passing through each $Y \in \mathcal{B}$. □

Proposition 4.6. *Let $X \subset \mathbb{P}^n$ be a reduced hypersurface of degree d over $\mathbb{F}_q$. Then the number of $\mathbb{F}_q$-hyperplanes $H \subset \mathbb{P}^n$ such that H does not intersect X properly or $X \cap H$ is non-reduced is bounded by the number*

$$(d-t)(d-1)(q+1)^2 + \frac{1}{2}t(t-1)(q+1) + 1 \;\leq\; d(d-1)(q+1)^2 + 1.$$

Proof. A hyperplane H as in the statement is of type (I), (II), or (III), so the bound on the left hand side is obtained by summing up the bounds in Lemmas 4.4 and 4.5. The inequality can be verified directly so we leave it to the reader. □

Proof of Theorem 4.1. By Proposition 4.6, we will get a desirable hyperplane if

$$\sum_{j=0}^{n} q^j > (d-t)(d-1)(q+1)^2 + \frac{1}{2}t(t-1)(q+1) + 1. \tag{4.3}$$

We can cancel the constant 1 on the right side by starting the sum on the left with $j=1$. In fact, we will ensure that a stronger inequality holds:

$$\sum_{j=1}^{n} q^j > \left((d-t)(d-1) + \frac{1}{2}t(t-1)\right)(q+1)^2. \tag{4.4}$$

We want to maximize the quantity

$$\phi(t) := (d-t)(d-1) + \frac{1}{2}t(t-1)$$

as a function of t on the interval $[0, d]$. Note that $\phi(t)$ is a quadratic polynomial in t with the leading term $(1/2)t^2$. As the graph of $\phi(t)$ is the usual upward-facing parabola, the maximum is attained at the end point $t=0$ or $t=d$. Since $\phi(0) = d(d-1)$ and $\phi(d) = \frac{1}{2}d(d-1)$, we conclude that $\phi(t) \leq d(d-1)$.

Straightforward computations show that the inequality

$$q^{n-3}(q-1) \geq d(d-1) \tag{4.5}$$

holds in the following cases:

- $n = 3$ and $q \geq d(d-1)+1$,
- $n \geq 4$ and $q \geq d$.

By hypothesis, we have $n \geq 3$, which implies that

$$\sum_{j=1}^{n} q^j > q^{n-3}(q^3+q^2-q-1) = q^{n-3}(q-1)(q+1)^2.$$

Combining this with (4.5), we obtain

$$\sum_{j=1}^{n} q^j > d(d-1)(q+1)^2 \geq \phi(t)(q+1)^2 = \left((d-t)(d-1)+\frac{1}{2}t(t-1)\right)(q+1)^2.$$

which is exactly (4.4), as desired. □

Remark 4.7. Note that inequality (4.3) fails when $n = 2$ and hence necessitates a different approach in Section 4.2.

While our main Theorem 4.1 is only stated for reduced hyperplane sections for the sake of simplicity, it easily extends by induction to the following more general result.

Corollary 4.8. *Let $X \subset \mathbb{P}^n$ be a reduced hypersurface of degree $d \geq 2$ over $\mathbb{F}_q$ with $n \geq 3$. Then, for every $2 \leq r \leq n-1$, there exists an r-plane $T \subset \mathbb{P}^n$ over $\mathbb{F}_q$ such that $X \cap T$ is reduced and has the expected dimension $r-1$ provided that*

$$q \geq d(d-1)+1.$$

The proof of the corollary is left as an easy exercise to the reader.

4.2. **Transverse lines to reduced plane curves.** We prove Theorem 4.2 in this section.

Lemma 4.9. *Suppose that $C \subset \mathbb{P}^2$ is an integral curve over $\overline{\mathbb{F}_q}$ of degree $d \geq 2$. Then the number of $\mathbb{F}_q$-lines not transverse to C is bounded by*

$$\frac{1}{2}(d-1)(3d-2)(q+1).$$

Proof. An $\mathbb{F}_q$-line L is not transverse to C either if it passes through a singular point of C or if $L = T_P C$ for some $P \in C$. Since C is irreducible, one can derive from, for example, [Liu02, §7.5, Proposition 5.4], that the number of singular points of C is at most

$$\frac{1}{2}(d-1)(d-2).$$

Because each singular point has at most $q+1$ distinct $\mathbb{F}_q$-lines passing through it, this accounts for

$$\frac{1}{2}(d-1)(d-2)(q+1)$$

many non-transverse $\mathbb{F}_q$-lines.

To estimate the number of the second type of non-transverse lines, first note that the condition $L = T_P C$ implies that $T_P C$ is defined over $\mathbb{F}_q$, thus $P \in Z_C$ by Proposition 3.1. As C is integral, it intersects some D_{ij} in 0-dimensional scheme by Lemma 4.3. Thus, the number of $\mathbb{F}_q$-lines that arise as $T_P C$ is at most

$$C \cdot D_{ij} = \deg(C)\deg(D_{ij}) = d(d-1)(q+1).$$

Consequently, the number of non-transverse $\mathbb{F}_q$-lines to C is at most

$$\frac{1}{2}(d-1)(d-2)(q+1) + d(d-1)(q+1) = \frac{1}{2}(d-1)(3d-2)(q+1)$$

as claimed. □

Lemma 4.10. *Let $C \subset \mathbb{P}^2$ be a reduced curve of degree $d \geq 2$ over $\mathbb{F}_q$. Then the number of $\mathbb{F}_q$-lines not transverse to C is bounded by*

$$\frac{3}{2}d(d-1)(q+1).$$

Proof. Write $C_{\overline{\mathbb{F}_q}} = \cup_{i=1}^{\ell} C_i$ where each C_i is irreducible and let $d_i := \deg(C_i)$. For each $\mathbb{F}_q$-line L not transverse to C, we have that

(i) L meets C_i non-transversely for some i where $\deg(C_i) \geq 2$, or
(ii) L passes through an intersection point of C_i and C_j for some $i \neq j$. Note that this includes L which meets C_i non-transversely where $\deg(C_i) = 1$.

By applying Lemma 4.9 to each component C_i and summing up all the upper bounds, we conclude that the number of lines in (i) is at most

$$\frac{1}{2}\sum_{i=1}^{\ell}(d_i-1)(3d_i-2)(q+1).$$

On the other hand, the number of points in $C_i \cap C_j$ for $i \neq j$ is at most $d_i d_j$ by Bézout's theorem. Since there are at most $(q+1)$ lines defined over $\mathbb{F}_q$ that pass through a point in $C_i \cap C_j$, the number of lines in (ii) is at most

$$\sum_{i<j} d_i d_j (q+1).$$

By adding up all the contributions above, we obtain that the number of $\mathbb{F}_q$-lines not transverse to C is at most

$$(q+1)\left(\frac{1}{2}\sum_{i=1}^{\ell}(d_i-1)(3d_i-2) + \sum_{i<j} d_i d_j\right)$$

As a factor of the above, we have

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{\ell}(d_i-1)(3d_i-2) + \sum_{i<j} d_i d_j &= \sum_{i=1}^{\ell}\left(\frac{3}{2}d_i^2 - \frac{5}{2}d_i + 1\right) + \sum_{i<j} d_i d_j \\
&= \sum_{i=1}^{\ell}\left(d_i^2 - \frac{5}{2}d_i + 1\right) + \frac{1}{2}\left(\sum_{i=1}^{\ell} d_i^2 + 2\sum_{i<j} d_i d_j\right) \\
&= \sum_{i=1}^{\ell}\left(d_i^2 - \frac{5}{2}d_i + 1\right) + \frac{1}{2}\left(\sum_{i=1}^{\ell} d_i\right)^2 \\
&= \left(\sum_{i=1}^{\ell} d_i^2\right) - \frac{5}{2}d + \ell + \frac{1}{2}d^2
\end{aligned}$$

where the last equality uses $d = \sum_{i=1}^{\ell} d_i$. Using the facts that

$$\sum_{i=1}^{\ell} d_i^2 \leq \left(\sum_{i=1}^{\ell} d_i \right)^2 = d^2 \quad \text{and} \quad \ell \leq d,$$

we conclude that the number of $\mathbb{F}_q$-lines not transverse to C is at most

$$\left(d^2 - \frac{5}{2}d + d + \frac{1}{2}d^2 \right)(q+1) = \left(\frac{3}{2}d^2 - \frac{3}{2}d \right)(q+1) = \frac{3}{2}d(d-1)(q+1)$$

as desired. □

Proof of Theorem 4.2. The number of $\mathbb{F}_q$-lines in $\mathbb{P}^2$ is q^2+q+1, so, by Lemma 4.10, there exists a transverse $\mathbb{F}_q$-line if

$$q^2 + q + 1 > \frac{3}{2}d(d-1)(q+1).$$

This inequality holds under the hypothesis $q \geq \frac{3}{2}d(d-1)$. Indeed, we have

$$q^2 + q + 1 > q^2 + q = q(q+1) \geq \frac{3}{2}d(d-1)(q+1)$$

which completes the proof. □

Remark 4.11. There is a different method [AG22, Proposition 2.2] to prove Theorem 4.2 at the cost of slightly stronger hypothesis $q \geq 2d(d-1)$.

Our final result in the present paper concerns existence of transverse $\mathbb{F}_q$-lines to reduced hypersurfaces of arbitrary dimension. We obtain it by reducing the statement to the case of plane curves (Theorem 4.2) with the help of Corollary 4.8.

Corollary 4.12. *Let $X \subset \mathbb{P}^n$ be a reduced hypersurface of degree d defined over $\mathbb{F}_q$. Suppose that*

$$q \geq \frac{3}{2}d(d-1).$$

Then there exists an $\mathbb{F}_q$-line $L \subset \mathbb{P}^n$ which is transverse to X.

Proof. When $d = 1$, X is a hyperplane and any $\mathbb{F}_q$-line $L \subset \mathbb{P}^n$ with $L \not\subset H$ satisfies the conclusion. For $d \geq 2$, it is straightforward to see that the inequality $q \geq \frac{3}{2}d(d-1)$ implies $q \geq d(d-1)+1$. By Corollary 4.8, there exists a plane $H \cong \mathbb{P}^2$ in $\mathbb{P}^n$ over $\mathbb{F}_q$ such that $X_1 := X \cap H$ is a reduced plane curve. Now we apply Theorem 4.2 to find an $\mathbb{F}_q$-line $L \subset \mathbb{P}^2$ such that $X_1 \cap L$ consists of d distinct points. This line L also satisfies the condition that $\#(X \cap L) = d$ distinct points, and so L is a desired transverse line to X. □

References

[ADL21] Shamil Asgarli, Lian Duan, and Kuan-Wen Lai, *Transverse lines to surfaces over finite fields*, Manuscripta Math. **165** (2021), no. 1-2, 135–157.

[AG22] Shamil Asgarli and Dragos Ghioca, *A Bertini type theorem for pencils over finite fields*, Finite Fields Appl. **77** (2022), Paper No. 101936.

[Asg19a] Shamil Asgarli, *Sharp Bertini theorem for plane curves over finite fields*, Canad. Math. Bull. **62** (2019), no. 2, 223–230.

[Asg19b] ———, *Transversality for Algebraic Varieties over Finite Fields*, Ph.D. Thesis, 2019.

[Bal03] Edoardo Ballico, *An effective Bertini theorem over finite fields*, Adv. Geom. **3** (2003), no. 4, 361–363.

[CMP15] Antonio Cafure, Guillermo Matera, and Melina Privitelli, *Polar varieties, Bertini's theorems and number of points of singular complete intersections over a finite field*, Finite Fields Appl. **31** (2015), 42–83.

[Cou16] Alain Couvreur, *An upper bound on the number of rational points of arbitrary projective varieties over finite fields*, Proc. Amer. Math. Soc. **144** (2016), no. 9, 3671–3685.

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[HK13] Masaaki Homma and Seon Jeong Kim, *An elementary bound for the number of points of a hypersurface over a finite field*, Finite Fields Appl. **20** (2013), 76–83.

[Kat99] Nicholas M. Katz, *Space filling curves over finite fields*, Math. Res. Lett. **6** (1999), no. 5-6, 613–624.

[Kle74] Steven L. Kleiman, *The transversality of a general translate*, Compositio Math. **28** (1974), 287–297.

[Kle86] ———, *Tangency and duality*, Proceedings of the 1984 Vancouver conference in algebraic geometry, 1986, pp. 163–225.

[KP91] Steven Kleiman and Ragni Piene, *On the inseparability of the Gauss map*, Enumerative algebraic geometry (Copenhagen, 1989), 1991, pp. 107–129.

[Liu02] Qing Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[MP93] Dragoslav S. Mitrinović and Josip E. Pečarić, *Bernoulli's inequality*, Rend. Circ. Mat. Palermo (2) **42** (1993), no. 3, 317–337 (1994).

[MPP16] Guillermo Matera, Mariana Pérez, and Melina Privitelli, *Explicit estimates for the number of rational points of singular complete intersections over a finite field*, J. Number Theory **158** (2016), 54–72.

[Poo04] Bjorn Poonen, *Bertini theorems over finite fields*, Ann. of Math. (2) **160** (2004), no. 3, 1099–1127.

[Zak93] F. L. Zak, *Tangents and secants of algebraic varieties*, Translations of Mathematical Monographs, vol. 127, American Mathematical Society, Providence, RI, 1993. Translated from the Russian manuscript by the author.

S. Asgarli, Department of Mathematics and Computer Science
Santa Clara University
Santa Clara, CA 95050, USA
sasgarli@scu.edu

L. Duan, Institute of Mathematical Sciences
ShanghaiTech University
No.393 Middle Huaxia Road, Pudong New District, Shanghai, China
duanlian@shanghaitech.edu.cn

K.-W. Lai, Institute of Mathematics, Academia Sinica
6F, Astronomy-Mathematics Building
No. 1, Sec. 4, Roosevelt Road, Taipei 106319, Taiwan
Email: kwlai@gate.sinica.edu.tw