A tour of the rationality problem of cubic fourfolds

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Abstract

The rationality problem in algebraic geometry studies whether a given variety admits a parametrization by a projective space, and how to construct a good one that faithfully reflects the geometry of the variety. Cubic fourfolds, i.e., smooth complex cubic hypersurfaces in $\mathbb{P}^5$, form the simplest type of examples whose rationality is still poorly understood. They have received much attention in recent years due to the simple construction as well as their intriguing relationship with K3 surfaces. The purpose of this note is to give an overview of this subject to a broad audience in algebraic geometry, with an emphasis on examples and how these examples inspired the formations of conjectures and recent developments.

1 Introduction

An algebraic variety X of dimension n defined over a field k is called *rational* if its function field is isomorphic to the field of rational functions in n independent variables, or equivalently, if there exists a birational map $X \overset{\sim}{\dashrightarrow} \mathbb{P}^n$. Otherwise, we call X *irrational*. Projective spaces are clearly rational. A quadratic hypersurface $Q \subset \mathbb{P}^{n+1}$ with a k-rational point $p \in Q$ is also rational as the stereographic projection from p maps Q birationally onto $\mathbb{P}^n$. For smooth cubic hypersurfaces defined over $\mathbb{C}$, it is known that:

- Every cubic curve $E \subset \mathbb{P}^2$ is an elliptic curve, thus is irrational.
- Every cubic surface $S \subset \mathbb{P}^3$ is rational as it is the blowup of $\mathbb{P}^2$ at six points. (See, for example, [Bea96, Theorem IV.13].)
- Every cubic threefold $Y \subset \mathbb{P}^4$ is irrational by Clemens and Griffiths [CG72]. They proved this by showing that the intermediate Jacobian of Y cannot be a product of Jacobians of curves, which must occur for a rational threefold.

For cubic fourfolds $X \subset \mathbb{P}^5$, it is expected that a very general one is irrational, through there is no example found so far. On the other hand, there exist some evidences indicating that certain type of cubic fourfolds are rational. To make this more precise, consider the Zariski open subset $U \subset |\mathcal{O}_{\mathbb{P}^5}(3)|$ of smooth cubic polynomials up to rescaling. Then the moduli space of cubic fourfolds is given as the quotient

$$\mathcal{C} := [U/\mathrm{PGL}_6(\mathbb{C})]$$

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in the sense of geometric invariant theory [MFK94, §4.2]. This is a Deligne–Mumford stack with a quasi-projective course moduli space of dimension 20. A cubic fourfold is called *special* if it contains an algebraic surface not homologous to a complete intersection. They form countably many irreducible divisors $\mathcal{C}_d \subset \mathcal{C}$ indexed by the integers

$$d \geq 8 \quad \text{and} \quad d \equiv 0, 2 \pmod 6. \tag{1.1}$$

Moreover, every $X \in \mathcal{C}_d$ can be associated with a K3 surface in terms of Hodge theory provided that, in addition,

$$d \text{ is not divisible by } 4, 9, \text{ or any odd prime } p \equiv 2 \pmod 3. \tag{1.2}$$

The rationality conjecture for cubic fourfolds, formulated originally by Kuznetsov [Kuz10, Conjecture 1.1] in terms of derived categories, states that *a cubic fourfold is rational if and only if it admits an associated K3 surface.*

This article consists of three parts. In Section 2, we discuss Pfaffian cubic fourfolds, namely, the cubic fourfolds that are defined by the Pfaffians of 6×6 matrices in linear polynomials. As examples of rational cubic fourfolds, their rationality and associated K3 surfaces can be constructed in explicit ways, and these constructions have inspired many of later developments in the subject. In Section 3, we review a global picture established by Hassett [Has00] which involves all cubic fourfolds. The notions of special cubic fourfolds, associated K3 surfaces, and the main ideas behind the numerical conditions (1.1) and (1.2) will be introduced here. In Section 4, we give an overview of the rationality conjecture, its motivation, and the evidences that have been discovered so far.

2 Pfaffian cubic fourfolds

Recall that the Pfaffian of a $2n \times 2n$ skew-symmetric matrix A is $\mathrm{pf}(A) = \sqrt{\det(A)}$. Pfaffian cubic fourfolds are smooth cubics $X \subset \mathbb{P}^5$ defined by equations of the form $\mathrm{pf}(A) = 0$ where A is a 6×6 matrix in linear forms of $\mathbb{P}^5$. The study of such cubics can be dated back to 1940s and has received the attentions from many experts involving Morin [Mor40], Fano [Fan43], Tregub [Tre84, Tre93], and Beauville–Donagi [BD85]. They appear as examples of rational cubic fourfolds whose rationality and associated K3 surfaces can be constructed in relatively simple and explicit ways. In the following, we will review these constructions as well as how these cubics distribute in the moduli space.

2.1 Construction and the associated K3 surfaces

Let V be a complex vector space of dimension 6 and consider the space $\wedge^2 V$ of bivectors. Then the degenerate bivectors in the space $\mathbb{P}(\wedge^2 V) \cong \mathbb{P}^{14}$ form a hypersurface $\mathrm{Pf}(V) \subset \mathbb{P}(\wedge^2 V)$ which is a cubic defined by the Pfaffian of a 6×6 skew-symmetric matrix in independent variables. The hypersurface $\mathrm{Pf}(V)$ is singular along the locus of bivectors of rank at most 2, which coincides with the Grassmannian $\mathrm{Gr}(2, V) \subset \mathbb{P}(\wedge^2 V)$ under the Plücker embedding. In particular, we have a filtration

$$\mathrm{Gr}(2, V) \subset \mathrm{Pf}(V) \subset \mathbb{P}\left(\wedge^2 V\right).$$

The same construction on the dual space V^* gives another filtration

$$\mathrm{Gr}(2, V^*) \subset \mathrm{Pf}(V^*) \subset \mathbb{P}\left(\wedge^2 V^*\right).$$

Notice that each Grassmannian has codimension 6 in the ambient $\mathbb{P}^{14}$. Hence, for a general 5-plane $H \subset \mathbb{P}(\wedge^2 V^*)$, the intersection

$$X := H \cap \mathrm{Pf}(V^*) \subset H \cong \mathbb{P}^5 \tag{2.1}$$

is a Pfaffian cubic fourfold. On the other hand, the collection of linear forms on $\mathbb{P}(\wedge^2 V^*)$ vanishing along H forms the dual 8-plane $H^\perp \subset \mathbb{P}(\wedge^2 V)$. Taking intersection with $\mathrm{Gr}(2, V)$ produces a smooth surface

$$S := H^\perp \cap \mathrm{Gr}(2, V) \subset H^\perp \cong \mathbb{P}^8. \tag{2.2}$$

Proposition 2.1. *The surface $S \subset \mathbb{P}^8$ has degree 14 and is simply connected with trivial canonical bundle, i.e., S a polarized K3 surface of degree 14.*

Sketch of proof. The surface $S \subset \mathbb{P}^8$ has degree 14 follows from the fact that $\mathrm{Gr}(2, V)$ has degree 14 under the Pücker embedding, which can be verified via the Schubert calculus. It is simply connected due to the fact that $\mathrm{Gr}(2, V)$ is simply connected and the Lefschetz hyperplane theorem. To compute the canonical bundle, let $\mathcal{U}$ denote the tautological bundle of $\mathrm{Gr}(2, V)$. Then the Pücker embedding corresponds to the line bundle $\mathcal{O}(1) = \wedge^2 \mathcal{U}^*$, and the result follows by computing that $K_{\mathrm{Gr}(2,V)} \cong \mathcal{O}(-6)$ and the adjunction formula. ☐

Remark 2.2. For a variety $Y \subset \mathbb{P}^n$, its projective dual $Y^* \subset (\mathbb{P}^n)^*$ is defined as the collection of hyperplanes tangent to Y, or more precisely,

$$Y^* = \overline{\{H \in (\mathbb{P}^n)^* \mid H \supset T_p Y \text{ for some smooth point } p \in Y\}}.$$

Moreover, we have $Y^{**} = Y$ by the reflexivity theorem. In our setting, the two varieties

$$\mathrm{Gr}(2, V) \subset \mathbb{P}\left(\wedge^2 V\right) \quad \text{and} \quad \mathrm{Pf}(V^*) \subset \mathbb{P}\left(\wedge^2 V^*\right)$$

appear as the projective duals to each other. (See, for example, [BC09, Proposition 1.5]) This provides the basis for Kusnetsov's homological projective duality which was used to show that the K3 category of the Pfaffian cubic X is equivalent to the derived category $\mathcal{D}^b(S)$ of the associated K3.

2.1.1 Fano variety of lines as hyperkähler fourfolds The Fano variety $F(X)$ of lines on X is a smooth fourfold. In [BD85], Beauville and Donagi constructed an isomorphism

$$S^{[2]} \xrightarrow{\sim} F(X) \tag{2.3}$$

where $S^{[2]}$ is the Hilbert scheme of length two subschemes on S. Let us review how this map is defined: Since S appears as a general linear section on $\mathrm{Gr}(2, V)$, every point $p + q \in S^{[2]}$ determines a subspace $\langle p, q \rangle \subset V$ of dimension 4 in the following ways:

- If p and q are distinct points on S, then they correspond to disjoint subspaces in V of dimension 2, which span the subspace $\langle p, q\rangle \subset V$.
- If $p+q$ represents a nonreduced subscheme of S, then it corresponds to an element $p \in \mathrm{Gr}(2,V)$ with a linear map $q\colon p \to V/p$ up to rescaling. In this case, we can define $\langle p, q\rangle \subset V$ as the preimage of $\mathrm{Im}\, q \subset V/p$ under the quotient $V \to V/p$.

The 2-forms in H that vanish along $\langle p, q\rangle$ form a linear subspace $\langle p, q\rangle^{\perp H}$. It turns out that this is a line contained in X. (See Lemma 2.3.) The map (2.3) is defined by sending $p+q$ to $\langle p, q\rangle^{\perp H}$. We refer the reader to [BD85, Proposition 5 (i)] for the proof that it is an isomorphism.

Lemma 2.3. *The linear subspace $\langle p, q\rangle^{\perp H} \subset H$ is a line contained in X.*

Proof. All of the 2-forms that vanish along $\langle p, q\rangle$ form an 8-plane $\langle p, q\rangle^{\perp \mathbb{P}(\wedge^2 V^*)}$ in $\mathbb{P}(\wedge^2 V^*)$ and we have $\langle p, q\rangle^{\perp H} = \langle p, q\rangle^{\perp \mathbb{P}(\wedge^2 V^*)} \cap H$. As $H = H^{\perp\perp}$, the subspace $\langle p, q\rangle^{\perp H}$ is cut out by the hyperplanes parametrized by $H^{\perp}$. Notice that p and q are chosen from $H^{\perp}$ and they correspond to hyperplanes passing through $\langle p, q\rangle^{\perp \mathbb{P}(\wedge^2 V^*)}$, so $H^{\perp}$ imposes $9-2=7$ independent conditions on $\langle p, q\rangle^{\perp \mathbb{P}(\wedge^2 V^*)}$. It follows that $\langle p, q\rangle^{\perp H}$ has codimension 7 in the 8-plane $\langle p, q\rangle^{\perp \mathbb{P}(\wedge^2 V^*)}$ and thus is a line. Moreover, the 2-forms vanishing along $\langle p, q\rangle$ are necessarily degenerate, so the line $\langle p, q\rangle^{\perp H}$ is contained in $H \cap \mathrm{Pf}(V^*) = X$. □

2.1.2 Relations among the Hodge structures The isomorphism $F(X) \cong S^{[2]}$ from (2.3) and the lattice structure on $H^2(S^{[2]}, \mathbb{Z})$ induced by the Beauville–Bogomolov–Fujiki form build up the isomorphisms

$$H^2(F(X),\mathbb{Z}) \cong H^2(S^{[2]},\mathbb{Z}) \cong \mathbb{Z}\delta \oplus_{\perp} H^2(S,\mathbb{Z}), \quad \delta\cdot\delta = -2. \tag{2.4}$$

Here 2δ corresponds to the divisor of $S^{[2]}$ that parametrizes nonreduced subschemes, and the pairing on $H^2(S,\mathbb{Z})$ coincides with the intersection product on S. On the other hand, the incidence relation

$$\begin{array}{ccc} Y := \{(x,\ell) \in X \times F(X) \mid x \in \ell\} & \xrightarrow{p_2} & F(X) \\ \downarrow{\scriptstyle p_1} & & \\ X & & \end{array}$$

determines the Abel–Jacobi map

$$\alpha\colon H^4(X,\mathbb{Z}) \longrightarrow H^2(F(X),\mathbb{Z}) : v \longmapsto p_{2*}p_1^* v$$

which is an isomorphism of abelian groups. Let $h \in H^2(X,\mathbb{Z})$ be the class of hyperplane sections and let $g \in H^2(F(X),\mathbb{Z})$ be the polarization coming from $F(X) \subset \mathrm{Gr}(2,6) \subset \mathbb{P}^{14}$ where the second inclusion is the Plücker embedding. Then $\alpha(h^2) = g$ [BD85, §3], and α restricts to an isomorphism between the primitive parts :

$$\alpha'\colon H^4(X,\mathbb{Z})_{\mathrm{prim}} \xrightarrow{\sim} H^2(F(X),\mathbb{Z})_{\mathrm{prim}}(-1)$$

which preserves the Hodge and lattice structures [BD85, Proposition 6]. Together with (2.4), this induces a saturated embedding

$$H^2(S,\mathbb{Z})_{\mathrm{prim}}(-1) \hookrightarrow H^4(X,\mathbb{Z})_{\mathrm{prim}}.$$

This makes S an example of a *Hodge-associated K3 surface* which will be defined in §3.3.

2.2 Approaches to the rationality

Here we exhibit a few methods about proving that every Pfaffian cubic fourfold is rational. One of them uses the construction in §2.1 directly. The others use the fact that such a cubic contains special algebraic surfaces.

2.2.1 Direct application of the construction This method was taken by Beauville and Donagi [BD85] to prove the rationality. From (2.1) and the fact that X is disjoint from $\mathrm{Gr}(2, V^*)$, we see that every point $[\varphi] \in X$ is represented by a linear map $\varphi\colon V \to V^*$ such that $\ker\varphi \subset V$ has dimension exactly 2. Consider the incidence variety $Z \subset \mathbb{P}(V) \times X$ defined by

$$\begin{array}{ccc} Z = \{([v], [\varphi]) \in \mathbb{P}(V) \times X \mid v \in \ker\varphi\} & \xrightarrow{\pi_2} & X \\ \downarrow{\scriptstyle \pi_1} & & \\ \mathbb{P}(V). & & \end{array}$$

Note that the fiber of each $[\varphi] \in X$ under π_2 is of the form

$$\pi_2^{-1}([\varphi]) = \mathbb{P}(\ker\varphi) \cong \mathbb{P}^1$$

which is a line in $\mathbb{P}(V)$ under the map π_1. Since π_1 is birational due to Lemma 2.4 below, the map π_2 induces a birational map from a general $\mathbb{P}^4 \subset \mathbb{P}(V)$ onto X.

Lemma 2.4. *The morphism $\pi_1\colon Z \longrightarrow \mathbb{P}(V)$ is birational.*

Proof. Let $v \in V$ be a nonzero vector. By linear algebra, the set of $[\varphi] \in \mathbb{P}(\wedge^2 V^*)$ such that $\varphi(v) = 0$ is a 9-plane L_v contained in $\mathrm{Pf}(V^*)$. Recall that $X = H \cap \mathrm{Pf}(V^*)$ where H is a 5-plane. Then L_v and H meet in a linear subspace within X of dimension at least $(9+5) - 14 = 0$. This implies that π_1 is surjective with connected fibers, which forces it to be birational as $\dim(Z) = \dim(\mathbb{P}(V)) = 5$. □

2.2.2 Using quintic del Pezzo surfaces A cubic fourfold X is Pfaffian if and only if it contains a quintic del Pezzo surface $T \subset X$ [Bea00, Proposition 9.2 (a)]. Recall that T is abstractly isomorphic to the blowup of $\mathbb{P}^2$ at four points in general positions, and its embedding into $\mathbb{P}^5$ is defined by the anti-canonical system. In the following, we say that a line $\ell \subset \mathbb{P}^5$ is *secant to* T if it intersects T in two points counted with multiplicity. Notice that every such line meets X at one and only one point outside T. Suppose that

(1) a general point $p \in \mathbb{P}^5$ lies on a line secant to T, and

(2) the family of lines secant to T is parametrized by a rational fourfold W.

Consider the fibration $\eta\colon \widetilde{W} \to W$ where $\eta^{-1}(p)$ is the secant line parametrized by $p \in W$. Then the above hypothesis realizes X as a rational section of η, which implies that X is birational to W and thus rational. In the following, we introduce one of the ways in establishing these conditions.

Lemma 2.5. *Let $\mathcal{I}_T$ be the ideal sheaf of $T \subset \mathbb{P}^5$. Then $h^0(\mathbb{P}^5, \mathcal{I}_T(2)) = 5$ and $\mathcal{I}_T(2)$ is generated by global sections.*

Proof. Using the short exact sequence $0 \longrightarrow \mathcal{I}_T \longrightarrow \mathcal{O}_{\mathbb{P}^5} \longrightarrow \mathcal{O}_T \longrightarrow 0$ and the Riemann–Roch formula for T, one can verify that

$$h^0(\mathbb{P}^5, \mathcal{I}_T(2)) = h^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2)) - h^0(T, \mathcal{O}_T(2)) = 21 - 16 = 5.$$

Similar computations imply $h^1(\mathbb{P}^5, \mathcal{I}_T(1)) = h^2(\mathbb{P}^5, \mathcal{I}_T) = 0$, so the Castelnuovo–Mumford regularity of $\mathcal{I}_T$ equals 2, whence $\mathcal{I}_T(2)$ is generated by global sections. $\square$

By Lemma 2.5, the linear system $|\mathcal{I}_T(2)|$ defines a rational map $f \colon \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ which can be resolved by a blowup along T, as depicted below

$$\begin{array}{ccc} Y = \mathrm{Bl}_T\mathbb{P}^5 & & \\ \downarrow & \searrow^{f'} & \\ \mathbb{P}^5 & \overset{f}{\dashrightarrow} & \mathbb{P}^4. \end{array}$$

Because f is defined by the quadrics passing through T, a line secant to T is contracted by this map. On the other hand, we have that

Proposition 2.6. *A general fiber of f appears as a line secant to T.*

Proof. In the Picard group of Y, let H (resp. H') denote the pullback of the hyperplane class on $\mathbb{P}^5$ (resp. $\mathbb{P}^4$) and E be the exceptional divisor of the blowup $Y \to \mathbb{P}^5$. Due to the construction of f, these classes satisfy the relation $H' = 2H - E$, and the class corresponds to a general fiber of f equals H'^4. We want to show that

$$HH'^4 = H(2H-E)^4 = 1 \quad \text{and} \quad EH'^4 = E(2H-E)^4 = 2 \tag{2.5}$$

as the former (resp. the latter) says that a general fiber is a line (resp. is secant to T). To compute these intersection numbers, first recall that the Segre class of $i \colon T \hookrightarrow \mathbb{P}^5$ is given by $s(T, \mathbb{P}^5) = c(N_{T/\mathbb{P}^5})^{-1} = c(T) \cdot i^*c(\mathbb{P}^5)^{-1}$. Let $h \in \mathrm{Pic}(T)$ denote the class of hyperplane sections. Then for each $k = 0, \ldots, 5$, a straightforward computation shows

$$H^{5-k}E^k = (-1)^{k-1} \int_T h^{5-k} \cdot s(T, \mathbb{P}^5) = \begin{cases} 1 & \text{if} \quad k = 0 \\ 0 & \text{if} \quad k = 1, 2 \\ 5 & \text{if} \quad k = 3 \\ 25 & \text{if} \quad k = 4 \\ 82 & \text{if} \quad k = 5 \end{cases}$$

Then the result follows by inserting these numbers back to (2.5). $\square$

Proposition 2.6 implies condition (1) immediately. It also implies that the family of secant lines of T is birationally parametrized by the codomain $\mathbb{P}^4$ of the map f, thus establishes condition (2). This concludes that X is rational. Notice that the strict transform of the cubic X to the blowup Y appears as a rational section of f'.

Remark 2.7. Let $\pi_p\colon T \to \mathbb{P}^4$ be the projection from a general point $p \in \mathbb{P}^5$. Using the double point formula [Ful98, Theorem 9.3], one can verify that the image $\pi_p(T)$ is singular along one double point, which implies that p lies on one and only one secant line of T, and thus establishes (1). Condition (2) can be confirmed by the fact that the Hilbert scheme of length two subschemes on a rational surface is rational. (See, for example, [ABCH13, §10.1].) This provides a slightly different approach to the rationality.

Remark 2.8. The idea presented in this section has been generalized by Russo and Staglianò to produce new examples of rational cubic fourfolds [RS19, RS20]. In their construction, secant lines are replaced by rational curves of higher degrees that intersect a surface in a cubic fourfold with higher multiplicities. We will review their results in §4.3.

2.2.3 Using quartic scrolls Let X be a Pfaffian cubic fourfold and S be its associated K3 surface. Recall from (2.3) that there is an isomorphism

$$F(X) \xrightarrow{\sim} S^{[2]}.$$

The locus on $S^{[2]}$ that parametrizes nonreduced subschemes is naturally isomorphic to the projective tangent bundle $\mathbb{P}(TS)$. Under the above isomorphism, the $\mathbb{P}^1$-fibers of this bundle correspond to rational curves $R \subset F(X)$, which then induce a family of quartic scrolls $\Sigma \subset X$ parametrized by S [HT01, Example 7.7]. The surface $\Sigma \subset \mathbb{P}^5$ is isomorphic to either

- $\mathbb{P}^1 \times \mathbb{P}^1$ embedded via the system $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2)|$, or
- the Hirzebruch surface $\mathbb{F}_2$ embedded via $|\mathcal{O}_{\mathbb{F}_2}(h+f)|$ where h (resp. f) is the class of a section (resp. of a fiber) so that $f^2 = 0$, $f \cdot h = 1$, and $h^2 = 2$.

The second case is a specialization of the first. (For this fact, we refer the reader to [Lai17, Proposition 4.1 (3)] for a general description about how rational scrolls of different types distribute in the Hilbert scheme.)

Proposition 2.9. *In both cases, the system of quadrics passing through Σ defines a rational map $g\colon \mathbb{P}^5 \dashrightarrow \mathbb{P}^5$ that can be resolved by blowing up Σ, as depicted below*

$$\begin{array}{ccc} Y := \mathrm{Bl}_\Sigma\mathbb{P}^5 & & \\ {\scriptstyle\epsilon}\downarrow & \overset{g'}{\searrow} & \\ \mathbb{P}^5 & \overset{g}{\dashrightarrow} & \mathbb{P}^5. \end{array}$$

Moreover, the image of g is a quadratic hypersurface $Q \subset \mathbb{P}^5$, and a general fiber of g is a line secant to Σ.

Sketch of proof. The first statement can be proved in the same way as Lemma 2.5. Here we refer the reader to [Lai17, Lemma 1.1] if they need data about sheaf cohomologies on Hirzebruch surfaces during their computation. The remaining part of the proposition can be verified using intersection theory as in Proposition 2.6. For example, to see that the image is a quadric $Q \subset \mathbb{P}^5$, we first denote by $H \in \mathrm{Pic}(Y)$ (resp. $H' \in \mathrm{Pic}(Y)$) the pullback of the hyperplane class in the domain of g (resp. the codomain of g) and let $E \in \mathrm{Pic}(Y)$ be the exceptional divisor of ϵ. Then $H' = 2H - E$. One can compute that $H'^5 = 0$ and $HH'^4 = 2$, where the former says that g is not dominant, and the latter implies that the image $g(\mathbb{P}^5) \subset \mathbb{P}^5$ is either

(i) a hyperplane and the preimage under g of a general line in $\mathbb{P}^5$ is a conic, or

(ii) a quadric and the preimage under g of a general line in $\mathbb{P}^5$ consists of two lines.

Condition (i) is impossible because it implies that the quadrics that define g are linearly dependent. Hence we are in condition (ii). □

Proposition 2.9 establishes conditions (1) and (2) in §2.2.2. In this case, the cubic fourfold X appears as a rational section of a $\mathbb{P}^1$-bundle over a quadric Q, where a general $\mathbb{P}^1$-fiber corresponds to a secant line of Σ. In particular, X is birational to Q and thus is rational.

2.3 Cubic fourfolds containing disjoint planes

Cubic fourfolds containing two disjoint planes appear as limits of Pfaffian cubic fourfolds in the moduli space ([Tre93], see also [Has16, Remark 7]). As an explicit example, consider $\mathbb{P}^5$ in homogeneous coordinates $[u : v : w : x : y : z]$ and the two planes

$$P_1 := \{x = y = z = 0\} \quad \text{and} \quad P_2 := \{u = v = w = 0\}.$$

Then the cubic hypersurface

$$X := \{ux^2 + vy^2 + wz^2 = u^2x + v^2y + w^2z\} \subset \mathbb{P}^5$$

is smooth and contains P_1 and P_2. Notice that a general point in $\mathbb{P}^5$ lies on a unique line joining P_1 and P_2, and a line joining P_1 and P_2 intersects X in a third point outside the two planes. (Compare these with (1) and (2) in §2.2.2.) This induces a birational map

$$\rho\colon P_1 \times P_2 \overset{\simeq}{\dashrightarrow} X : (a, b) \longmapsto \rho(a, b) \quad \text{where} \quad \langle a, b\rangle \cap X = \{a, b, \rho(a, b)\}.$$

The construction of the associated K3 surface is very straightforward in this case. Indeed, the map ρ is undefined if and only if the line spanned by the pair $(a, b) \in P_1 \times P_2$ is contained in X, so the base locus of ρ is defined by

$$\mathrm{Bs}(\rho) = \{ux^2 + vy^2 + wz^2 = 0 = u^2x + v^2y + w^2z\} \subset P_1 \times P_2.$$

This is a complete intersection in $\mathbb{P}^2 \times \mathbb{P}^2$ cut out by polynomials of bidegrees $(1, 2)$ and $(2, 1)$, so it is a K3 surface.

Remark 2.10. Pfaffian cubic fourfolds are Zariski dense in an irreducible divisor $\mathcal{C}_{14}$ in the moduli space. They are examples of *special cubic fourfolds* to be defined in §3.2. The locus $\mathcal{C}_{14}$ consists of special cubic fourfolds X labelled by a rank two saturated sublattice

$$h^2 \in \begin{pmatrix} 3 & 5 \\ 5 & 13 \end{pmatrix} \cong \begin{pmatrix} 3 & 4 \\ 4 & 10 \end{pmatrix} \subset H^{2,2}(X, \mathbb{Z}) \quad \text{where} \quad h = c_1(\mathcal{O}_X(1)). \tag{2.6}$$

When X contains a quintic del Pezzo surface T (resp. a quartic scroll Σ), the lattice $\langle h^2, T\rangle$ (resp. $\langle h^2, \Sigma\rangle$) has the intersection matrix on the left (resp. on the right). Consider the following subsets of $\mathcal{C}_{14}$:

$$\begin{aligned}\mathcal{C}_{\mathrm{Pf}} &= \{X \in \mathcal{C}_{14} \mid X \text{ is Pfaffian}\}\\ \mathcal{C}_{\mathrm{dP}} &= \{X \in \mathcal{C}_{14} \mid X \text{ contains a quintic del Pezzo surface}\}\\ \mathcal{C}_{\mathrm{RS}} &= \{X \in \mathcal{C}_{14} \mid X \text{ contains a quartic rational scroll}\}\\ \mathcal{C}_{\Pi} &= \{X \in \mathcal{C}_{14} \mid X \text{ contains disjoint planes}\}\end{aligned}$$

The first 3 subsets contain a Zariski open subset in $\mathcal{C}_{14}$, and we have $\mathcal{C}_{\mathrm{Pf}} = \mathcal{C}_{\mathrm{dP}} \subset \mathcal{C}_{\mathrm{RS}}$. However, it was proved by Bolognesi, Russo, and Staglianò [BRS19, Theorem 3.7] that $\mathcal{C}_{\mathrm{Pf}}$ is *not* Zariski open in $\mathcal{C}_{14}$. On the other hand, $\mathcal{C}_{\Pi}$ has codimension one in $\mathcal{C}_{14}$, and Auel [Aue20, Theorem 1] proved that the complement $\mathcal{C}_{14} \setminus \mathcal{C}_{\mathrm{Pf}}$ is contained in $\mathcal{C}_{\Pi}$. In particular, a member of $\mathcal{C}_{14}$ is Pfaffian or contains disjoint planes (or both).

3 Hodge theory of special cubic fourfolds

In this section, we will review the definitions of special cubic fourfolds and associated K3 surfaces, and try to give the ideas about how the numerical conditions (1.1) and (1.2) are obtained. The main references for this part are Hassett's seminal paper [Has00] and his lecture note [Has16].

3.1 Hodge structures of cubic fourfolds

3.1.1 Middle cohomologies as lattices The Hodge diamond of a cubic fourfold has the shape

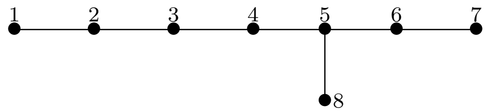
$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&0&&0&&& \\ &&0&&1&&0&& \\ &0&&0&&0&&0& \\ 0&&1&&21&&1&&0. \end{array}$$

Notice that $H^4(X,\mathbb{Z})$ has a weight two Hodge structure. These Hodge numbers, together with the Riemann bilinear relations, imply that the middle cohomology $H^4(X,\mathbb{Z})$ equipped with the intersection product is a unimodular lattice of signature $(21,2)$. Let $h \in H^2(X,\mathbb{Z})$ denote the class of hyperplane sections. Then the square $h^2 \in H^4(X,\mathbb{Z})$ has self-intersection $h^2 \cdot h^2 = h^4 = 3$. As every indefinite unimodular lattice carrying a vector of odd self-intersection possesses an orthogonal basis (see, for example, [MH73, Theorem 4.3]), the lattice $H^4(X,\mathbb{Z})$ is isomorphic to $\langle 1\rangle^{\oplus 21} \oplus \langle -1\rangle^{\oplus 2}$.

For the same reason, we can further identify the middle cohomology as

$$H^4(X,\mathbb{Z}) \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus \langle 1\rangle^{\oplus 3} \quad \text{where} \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.1}$$

Here E_8 is the unique unimodular even lattice of signature $(8, 0)$, which is spanned by the vectors $\{e_1, \ldots, e_8\}$ with $e_i \cdot e_i = 2$ for all i and $e_i \cdot e_j = -1$ for i-th and j-th vertices adjacent in the corresponding Dynkin diagram



Under the identification (3.1), we can let $h^2 = (1, 1, 1) \in \langle 1 \rangle^{\oplus 3}$ upon possibly taking a lattice automorphism. This allows one to compute explicitly the primitive cohomology as

$$H^4(X, \mathbb{Z})_{\text{prim}} = \langle h^2 \rangle^{\perp H^4(X,\mathbb{Z})} \cong U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2 \quad \text{where} \quad A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Notice that this is an even lattice of signature $(20, 2)$.

3.1.2 Torelli theorem for cubic fourfolds The Torelli theorem lays down the foundation for a description about the moduli spaces of *special cubic fourfolds* to be introduced in §3.2. We briefly review the theorem here.

For the sake of simplicity, let us denote

$$\Lambda := U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2.$$

The dual lattice $\Lambda^\vee := \operatorname{Hom}(\Lambda, \mathbb{Z})$ is the space of $\mathbb{Z}$-valued linear functionals on Λ. Due to the nondegeneracy of the pairing on Λ, we can identify

$$\Lambda^\vee = \{v \in \Lambda \otimes \mathbb{Q} \mid v \cdot w \in \mathbb{Z} \text{ for all } w \in \Lambda\}.$$

In particular, $\Lambda^\vee$ carries a pairing inherited from Λ, and we can consider Λ as a sublattice of $\Lambda^\vee$ via the natural embedding $\Lambda \to \Lambda^\vee : v \mapsto v \otimes 1$. The *discriminant group* of of Λ is defined as the quotient $\Lambda^\vee/\Lambda$, which can be computed explicitly via the isomorphisms

$$\Lambda^\vee/\Lambda \cong A_2^\vee/A_2 \cong \mathbb{Z}/3\mathbb{Z}.$$

Note that, as Λ is even, its pairing induces a $\mathbb{Q}/2\mathbb{Z}$-valued pairing on $\Lambda^\vee/\Lambda$.

The automorphisms of the lattice Λ arising from the monodromy of cubic fourfolds form the subgroup

$$\Gamma := \{g \in \mathrm{O}(\Lambda) \mid g \text{ acts trivially on } \Lambda^\vee/\Lambda\}.$$

This group acts on the *period domain* Ω, defined as one of the connected components of the open quadric $\{x \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid x \cdot x = 0,\ x \cdot \overline{x} > 0\}$. The *period space* of Λ is defined as the quotient $\mathcal{P} := \Gamma \backslash \Omega$. This is a 20-dimensional quasi-projective variety according to Baily–Borel [BB66]. Voisin's Torelli theorem [Voi86] asserts that the period map

$$\tau \colon \mathcal{C} \longrightarrow \mathcal{P} : X \longmapsto H^{1,3}(X, \mathbb{C}) \tag{3.2}$$

is an open immersion. In fact, the domain of this map can be extended to cover singular cubics. More precisely, cubic hypersurfaces with at worst an ordinary double point are stable in the sense of geometric invariant theory, which can be proved via the numerical criterion for stability [MFK94, §2.1] and the methods in [MFK94, §4.2]. Therefore, we can extend the quotient $\mathcal{C} = [U/\mathrm{PGL}_6(\mathbb{C})]$ to

$$\widetilde{\mathcal{C}} \supset \mathcal{C} \tag{3.3}$$

such that the complement $\widetilde{\mathcal{C}} \setminus \mathcal{C}$ parametrizes cubics with an ordinary double point.

Theorem 3.1 ([Voi86]). *The period map* (3.2) *extends to an open immersion*

$$\widetilde{\tau}\colon \widetilde{\mathcal{C}} \longrightarrow \mathcal{P} : X \longmapsto H^{1,3}(X,\mathbb{C})$$

where $H^{1,3}(X,\mathbb{C})$ stands for the limiting Hodge structure for singular X.

3.2 Special cubic fourfolds and their moduli

The integral Hodge conjecture is valid for cubic fourfolds [Voi13], which asserts that the sublattice

$$H^{2,2}(X,\mathbb{Z}) := H^{2,2}(X,\mathbb{C}) \cap H^4(X,\mathbb{Z})$$

for every cubic fourfold X is generated by the classes of algebraic surfaces. This lattice is spanned by h^2 if $X \in \mathcal{C}$ is *very general*, i.e., is away from the union of a countable collection of Zariski closed subsets. We say X is *special* if X contains an algebraic surface not homologous to a complete intersection. In this case, X carries at least one *labelling*, namely a rank two saturated sublattice $K \subset H^{2,2}(X,\mathbb{Z})$ that contains h^2. Suppose that K is spanned by h^2 and the class v of some algebraic surface. Then the *discriminant* of K is defined as

$$\mathrm{disc}(K) := \det \begin{pmatrix} h^2 \cdot h^2 & h^2 \cdot v \\ v \cdot h^2 & v \cdot v \end{pmatrix}$$

which is independent of the choices of basis elements. Using the lattice structures of the middle cohomology of X, one can verify that the discriminant d of a labelling is necessarily positive and satisfies $d \equiv 0$ or $2 \pmod 6$ [Has00, Proposition 3.2.2].

3.2.1 Irreducibility of the moduli spaces The locus $\mathcal{C}_d \subset \mathcal{C}$ of special cubic fourfolds possessing a labelling of discriminant d is an irreducible divisor provided that it is nonempty [Has00, Theorem 3.2.3]. To see this, let us define $\mathcal{P}_d \subset \mathcal{P}$ be the Γ-quotient of the union of hyperplane sections

$$\bigcup_{\substack{\text{labelling } K \subset H^4(X,\mathbb{Z}) \\ \text{of discriminant } d}} K^{\perp} \subset \Omega.$$

Then $\mathcal{C}_d = \tau^{-1}(\mathcal{P}_d)$ where τ is the period map (3.2). It is straightforward to verify that the action of Γ can be extended to the full $H^4(X,\mathbb{Z})$ by fixing h^2. This action acts transitively on the set of labellings with the same discriminant [Has00, Proposition 3.2.4]. For example, if d is not divisible by 9, one can verify that an arbitrary labelling of discriminant d lies in the orbit of the labelling

$$\begin{pmatrix} 3 & 0 \\ 0 & 2n \end{pmatrix} \text{ if } d = 6n \quad \text{and} \quad \begin{pmatrix} 3 & 1 \\ 1 & 2n+1 \end{pmatrix} \text{ if } d = 6n+2.$$

It follows that $\mathcal{P}_d$ is irreducible, hence $\mathcal{C}_d$ is irreducible.

3.2.2 Existence of special cubic fourfolds Hassett proved that $\mathcal{C}_d$ is nonempty for each $d \geq 8$ that satisfies $d \equiv 0$ or $2 \pmod 6$ [Has00, Theorem 4.3.1]. This was achieved via a correspondence between the set of cubics with an ordinary double point and the set of sextic K3 surfaces: Let $X_0 \subset \mathbb{P}^5$ be a cubic hypersurface singular along an ordinary double point $p \in X_0$. The projection from p defines a birational map $\pi_p \colon X_0 \overset{\sim}{\dashrightarrow} \mathbb{P}^4$ which factors as

$$\begin{array}{ccccc} & & \overline{X_0} = \mathrm{Bl}_S\mathbb{P}^4 & & \\ & \swarrow^{q_1} & & \searrow^{q_2} & \\ X_0 & & \overset{\pi_p}{\underset{\sim}{\dashrightarrow}} & & \mathbb{P}^4. \end{array} \tag{3.4}$$

The map q_1 is the blowup of X_0 at p. On the other hand, q_2 is the contraction of the lines in X_0 passing through p, which is the same as the blowup of $\mathbb{P}^4$ at a sextic K3 surface $S \subset \mathbb{P}^4$. To verify this, one can work with affine coordinates $(x_1, \ldots, x_5)$ with $p = (0, 0, 0, 0, 0)$ and express X_0 as the zero locus of $f_2 + f_3$, where f_2 and f_3 are homogeneous polynomials of degrees 2 and 3 respectively. Then S arises as the complete intersection $\{f_2 = f_3 = 0\}$ in $\mathbb{P}^4$. Conversely, given a sextic K3 surface $S \subset \mathbb{P}^4$, one obtains a cubic $X_0 \subset \mathbb{P}^5$ singular along an ordinary double point via the same diagram, where the inverse π_p^{-1} is defined by the system of cubics passing through S.

The above picture establishes an isomorphism between the complement $\mathcal{C}_6 := \widetilde{\mathcal{C}} \setminus \mathcal{C}$ and the period space of sextic K3 surfaces. To construct a cubic fourfold with a labelling of discriminant $d \geq 8$ with $d \equiv 0, 2 \pmod 6$, one start with a sextic K3 surface S such that the Picard group of S contains the lattice

$$\begin{pmatrix} 6 & 0 \\ 0 & -2n \end{pmatrix} \text{ if } \ d = 6n \quad \text{and} \quad \begin{pmatrix} 6 & 2 \\ 2 & -2n \end{pmatrix} \text{ if } \ d = 6n + 2.$$

This produces an $X_0 \in \mathcal{C}_6$ carrying a labelling of discriminant d via the construction (3.4), and a member of $\mathcal{C}_d$ can be obtained by smoothing X_0. We refer the reader to [Has00, §4.3] for the details.

Theorem 3.2 ([Has00, Theorems 3.2.3 and 4.3.1])**.** *The locus $\mathcal{C}_d \subset \mathcal{C}$ of special cubic fourfolds of discriminant d is an irreducible divisor, which is nonempty if and only if $d \geq 8$ and $d \equiv 0, 2 \pmod 6$.*

3.3 Associated K3 surfaces

There are two different ways to define how a K3 surface is associated with a given special cubic fourfold. The first one was introduced by Hassett in terms of of Hodge theory. The other one was introduced by Kuznetsov via derived categories. We review the Hodge theoretic definition here and leave the derived categorical one to §4.1.

Definition 3.3 (Associated K3 surfaces)**.** Let X be a special cubic fourfold with a labelling K of discriminant d and let S be a K3 surface with a polarization f of degree d. We say X and S are *(Hodge-)associated* if there exists an isomorphism

$$H^4(X, \mathbb{Z}) \supset K^{\perp} \xrightarrow{\sim} f^{\perp} \subset H^2(S, \mathbb{Z})(-1) \tag{3.5}$$

that preserves the Hodge structures.

3.3.1 Existence of associated K3 surfaces It turns out that the existence of an associated K3 surface is a lattice theoretic problem. Indeed, it is easy to see that the isomorphism (3.5) induces a lattice isomorphism between $K^\perp(-1)$ and the lattice

$$M_d := \langle -d\rangle \oplus U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}$$

underlying the primitive cohomology of a degree d K3 surface. Conversely, if there exists a lattice isomorphism $K^\perp(-1) \cong M_d$, then the surjectivity of the period map for K3 surfaces [Bea85, Siu81] asserts that there exists a pseudo-polarized K3 surface (S, f) with $f^2 = d$ such that there is an isomorphism (3.5) preserving the Hodge structures. Recall that f is a *pseudo-polarization* means that it lies in the closure of the Kähler cone of S. Then the smoothness of X implies that $K^\perp$ contains no vector of self-intersection 2 [Voi86, §4, Proposition 1]. This implies that S contains no (-2)-curve orthogonal to f, so f is a polarization. As a result, the searching for associated K3 surfaces boils down to a computation of lattices as treated in [Has00, Proposition 5.1.4].

Theorem 3.4 ([Has00, Theorem 5.1.3])**.** *A member $X \in \mathcal{C}_d$ admits an associated K3 surface if and only if d is not divisible by 4, 9, or any odd prime $p \equiv 2 \pmod 3$.*

3.3.2 Counting associated K3 surfaces For a very general $X \in \mathcal{C}_d$ with such a d, there exists exactly one associated K3 surface if $d \equiv 2 \pmod 6$ and exactly two if $d \equiv 0 \pmod 6$. This statement was established by considering the moduli space of marked cubic fourfolds

$$\mathcal{C}_d^{\mathrm{mar}} := \{(X, \iota\colon K \hookrightarrow H^{2,2}(X,\mathbb{Z})) \mid \iota(K) \text{ is a labelling of } X\}.$$

There exists no nontrivial automorphism of K that fixes h^2 if $d \equiv 2 \pmod 6$. When $d \equiv 0 \pmod 6$, the only nontrivial such automorphism acts on $(h^2)^{\perp K}$ as the multiplication by -1. This shows that the forgetting map $\mathcal{C}_d^{\mathrm{mar}} \to \mathcal{C}_d$ is a birational morphism for $d \equiv 2 \pmod 6$ and generically 2-to-1 for $d \equiv 0 \pmod 6$. Moreover, a comparison between the constructions of $\mathcal{C}_d^{\mathrm{mar}}$ and the period space of polarized K3 surfaces of degree d shows that there exists an embedding of the former into the latter [Has00, Corollary 5.2.4]. This yields:

Theorem 3.5 ([Has16, Proposition 24])**.** *Let $\mathcal{F}_d$ be the moduli space of polarized K3 surfaces of degree d. Then the assignment of a K3 surface to its associated cubic fourfold defines a rational map $\mathcal{F}_d \dashrightarrow \mathcal{C}_d$, which is birational if $d \equiv 2 \pmod 6$ and 2-to-1 if $d \equiv 0 \pmod 6$.*

4 K3 categories and the rationality conjecture

In this section, we review the formulation of the conjecture, the motivation, and the rational examples discovered up to the point when this article was written.

4.1 K3 categories of cubic fourfolds

Definition 4.1. Let $\mathcal{D}$ a k-linear triangulated category where k is an algebraically closed field of characteristic zero.

- A *semiorthogonal decomposition* $\mathcal{D} = \langle \mathcal{D}_1, \ldots, \mathcal{D}_m \rangle$ is an ordered collection of full triangulated subcategories $\mathcal{D}_1, \ldots, \mathcal{D}_m$ of $\mathcal{D}$ such that $\mathrm{Hom}(\mathcal{D}_i, \mathcal{D}_j) = 0$ for $i > j$ and for every object $F \in \mathcal{D}$ there exists a chain of morphisms

$$0 = F_m \longrightarrow F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 = F$$

  such that $\mathrm{cone}(F_i \longrightarrow F_{i-1}) \in \mathcal{D}_i$ for $1 \leq i \leq m$.

- An object $E \in \mathcal{D}$ is called *exceptional* if $\mathrm{Hom}(E, E) \cong k$ and $\mathrm{Ext}^p(E, E) = 0$ for $p \neq 0$. An ordered collection of exceptional objects $E_1, \ldots, E_m$ is called *exceptional* if $\mathrm{Ext}^p(E_i, E_j) = 0$ for $i > j$ and all $p \in \mathbb{Z}$.

In our setting, we have $\mathcal{D} = \mathcal{D}^b(X)$, the bounded derived category of coherent sheaves on a cubic fourfold $X \subset \mathbb{P}^5$. Note that $K_X \cong \mathcal{O}_X(-3)$ by the adjunction formula. Then a straightforward computation shows that

$$\mathrm{Ext}^p(\mathcal{O}_X, \mathcal{O}_X(-t)) \cong \begin{cases} \mathbb{C} & \text{for} \quad t = p = 0 \\ 0 & \text{for} \quad 0 < t \leq 2 \quad \text{and} \quad p \in \mathbb{Z} \end{cases}$$

where the second row can be verified with the aid of the Kodaira vanishing theorem. This shows that the triple $(\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2))$ form an exceptional collection. The *K3 category of* X is defined as the right orthogonal

$$\begin{aligned} \mathcal{A}_X &= \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle^{\perp} \\ &= \{F \in \mathcal{D}^b(X) \mid \mathrm{Hom}(G, F) = 0 \text{ for all } G \in \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle\} \\ &= \{F \in \mathcal{D}^b(X) \mid \mathrm{Ext}^*(\mathcal{O}_X, F(-t)) = 0 \text{ for } 0 \leq t \leq 2\}. \end{aligned}$$

Via [MS19, Proposition 2.4], we see that $\mathcal{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle$ is a semiorthogonal decomposition.

Definition 4.2 (D-associated K3 surfaces)**.** A cubic fourfold X admits a (D-)associated K3 surface S if the K3 category $\mathcal{A}_X$ is equivalent to the derived category $\mathcal{D}^b(S)$.

Addington and Thomas [AT14] conjectured that the conditions for the existence of an associated K3 surface in the senses of Hodge theory (Definition 3.3) and of derived categories should be equivalent. They proved that this is generically true:

Theorem 4.3 ([AT14, Theorem 1.1])**.** *A cubic fourfold X admits a D-associated K3 surface implies that it admits a Hodge-associated K3 surface, i.e., belongs to $\mathcal{C}_d$ for some d not divisible by* 4, 9 *and any odd prime* $p \equiv 2 \pmod 3$. *Conversely, for every such d, there exists a Zariski open subset in $\mathcal{C}_d$ where each member admits a D-associated K3 surface.*

4.2 The rationality conjecture and motivation

Hassett [Has99, Theorem 4.2] found a locus $\mathcal{C}_\delta$ of codimension one in $\mathcal{C}_8$ which parametrizes rational cubic fourfolds. Notice that a general member of $\mathcal{C}_8$ has no associated K3 surface due to Theorem 3.4. Nevertheless, Kuznetsov [Kuz10, Theorem 4.3] observed that a general $X \in \mathcal{C}_8$ admits an associated *twisted* K3 surface S in the sense that

$$\mathcal{A}_X \cong \mathcal{D}^b(S, \alpha)$$

where $\mathcal{D}^b(S, \alpha)$ is the bounded derived category of coherent sheaves on S twisted by the Brauer class $\alpha \in \mathrm{Br}(X) := H^2_{\text{ét}}(S, \mathbb{G}_m)$. Moreover, the class α is trivial if and only if X belongs to $\mathcal{C}_\delta$ [Kuz10, Proposition 4.7]. This fact, together with Hassett's result, motivates the following conjecture:

Conjecture 4.4 ([Kuz10, Conjecture 1.1])**.** *A cubic fourfold is rational if and only if it admits an associated K3 surface.*

In the following, let us review the geometric construction behind the above facts.

4.2.1 Cubic fourfolds containing a plane Let $X \subset \mathbb{P}^5$ be a cubic fourfold containing a plane P and let $h \in \mathrm{Pic}(X)$ denote the class of hyperplane sections. Then the classes h^2 and $[P]$ in $H^4(X, \mathbb{Z})$ span a sublattice with intersection matrix

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

This is a labelling of discriminant 8, thus $X \in \mathcal{C}_8$. As all planes in $\mathbb{P}^5$ are projectively equivalent, this shows that every cubic fourfold containing a plane belongs to $\mathcal{C}_8$.

Proposition 4.5. *A cubic fourfold $X \subset \mathbb{P}^5$ contains a plane if and only if $X \in \mathcal{C}_8$.*

Proof. It remains to prove the converse. In fact, this is a consequence of the numerical characterization given by Voisin [Voi86, §3]. Here we provide a proof based on the irreducibility of $\mathcal{C}_8$ [Has00, Theorem 3.2.3]. Let I_P denote the ideal sheaf of a plane $P \subset \mathbb{P}^5$. Then a straightforward computation shows that $|I_P(3)|$ has dimension 45 and the stabilizer of P in $\mathrm{PGL}_6(\mathbb{C})$ has dimension 26. It follows that the moduli of cubic fourfolds with a plane has dimension $45 - 26 = 19$, so there exists a Zariski open subset in $\mathcal{C}_8$ parametrizing such cubics. This implies that every member of $\mathcal{C}_8$ contains a plane because a plane can only deform to a plane in $\mathbb{P}^5$. □

The projection of $\mathbb{P}^5$ from a plane P defines a rational map onto $\mathbb{P}^2$ fibered in the 3-planes containing P. Each of these 3-planes cut out on X the union of P with a quadric surface. Therefore, the restriction of the projection defines a rational map $f\colon X \dashrightarrow \mathbb{P}^2$ fibered in the residual quadric surfaces. Resolving f by blowing up at P gives the diagram

$$\begin{array}{ccc} \mathcal{Q} := \mathrm{Bl}_P X & & \\ {\scriptstyle \epsilon}\downarrow & \searrow^{f'} & \\ X & \overset{f}{\dashrightarrow} & \mathbb{P}^2 \end{array} \tag{4.1}$$

where $f'\colon \mathcal{Q} \to \mathbb{P}^2$ is a morphism fibered in quadratic surfaces. Notice that the generic fiber of $\mathcal{Q}$ can be considered as a quadric surface defined over the function field $\mathbb{C}(\mathbb{P}^2)$.

Proposition 4.6. *Let $\mathcal{Q} \subset \mathbb{P}^n$ be a smooth quadratic hypersurface over an arbitrary field k. Then the following statements are equivalent:*

(1) *$\mathcal{Q}$ is rational.*

(2) *$\mathcal{Q}$ has a k-rational point.*

(3) *$\mathcal{Q}$ has a k'-rational point where k'/k is an extension of odd degree.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial. The converse (2) $\Rightarrow$ (1) holds as the projection from a k-rational point $p \in \mathcal{Q}$ defines a birational map $\mathcal{Q} \dashrightarrow \mathbb{P}^{n-1}$ over k. The other converse (3) $\Rightarrow$ (2) follows from Springer's theorem, see [EKM08, Corollary 18.5]. This completes the proof.

Let us remark that an elegant proof for (3) $\Rightarrow$ (2) when $\mathcal{Q}$ is a surface can be found in [Has99, Proposition 2.1]. In this case, the Fano scheme $F(\mathcal{Q})$ of lines on $\mathcal{Q}$ consists of two copies of $\mathbb{P}^1$, and the Abel–Jacobi map $\mathrm{CH}_0(\mathcal{Q}) \longrightarrow \mathrm{CH}_0(F(\mathcal{Q}))$ assigns to a 0-cycle the sum of the lines passing through it. Under this map, a 0-cycle on of odd degree on $\mathcal{Q}$ corresponds to a 0-cycle on $F(\mathcal{Q})$ with odd degree on each of the $\mathbb{P}^1$-components, which can then be twisted to be of degree one on each $\mathbb{P}^1$ by the canonical class. The last cycle represents two lines on $\mathcal{Q}$ which belong to different families of rulings, so their intersection is a rational point. □

By Proposition 4.6, the generic fiber of $\mathcal{Q}$ is birational to the projective plane over $\mathbb{C}(\mathbb{P}^2)$ if and only if it has a $\mathbb{C}(\mathbb{P}^2)$-rational point of odd degree. In other words, $\mathcal{Q}$ itself, and thus the cubic fourfold X, is rational if and only if $\mathcal{Q}$ has a multisection of odd degree.

Lemma 4.7. *Let $W \subset X$ be a surface. Then the degree of its strict transform on $\mathcal{Q}$ as a multisection of $\mathcal{Q} \to \mathbb{P}^2$ equals the number*

$$\delta(W) := \deg(W) - [W] \cdot [P].$$

Proof. In the Picard group of $\mathcal{Q}$, let ℓ be the pullback by f' the class of a line in $\mathbb{P}^2$ and e be the exceptional divisor of the blowing-up $\epsilon\colon \mathcal{Q} \to X$. Then we have $\ell = \epsilon^* h - e$, and the class of a fiber of $\mathcal{Q} \to \mathbb{P}^2$ is given by

$$\ell^2 = (\epsilon^* h - e)^2 = \epsilon^* h^2 - 2(\epsilon^* h)e + e^2.$$

Then the degree of W as a multisection equals

$$\epsilon^*[W] \cdot \ell^2 = [W] \cdot h^2 - 0 + \epsilon^*[W] \cdot e^2 = \deg(W) - [W] \cdot [P]$$

as desired. □

Theorem 4.8 ([Has99])**.** *If a cubic fourfold X contains a plane P and a surface W with odd $\delta(W)$, then it is rational. Such cubics are parametrized by a locus $\mathcal{C}_\delta$ of a countably infinite union of divisors in $\mathcal{C}_8$.*

Proof. The first statement follows from Lemma 4.7 and the description right before the lemma. For the second statement, observe that $\delta(h^2) = 3 - 1 = 2$ and $\delta(P) = 1 - 3 = -2$, hence a very general member of $\mathcal{C}_8$ does not contain a surface W with $\delta(W)$ odd. Then the result follows from [Has16, Proposition 12] and the existence of cubic fourfolds containing disjoint planes. □

4.2.2 Associated twisted K3 surfaces Here we review the construction of the twisted K3 surface associated to X. We will focus only on the geometric aspects and leave the technical details about derived categories to [Kuz10].

Lemma 4.9. *Suppose that $X \in \mathcal{C}_8$ is general. Then the fiber quadrics of $f\colon X \dashrightarrow \mathbb{P}^2$ are degenerate over a smooth sextic curve $C \subset \mathbb{P}^2$.*

Proof. This is well known and one can find a proof, e.g., in [Kuz10, Lemma 4.1]. Here we take an elementary approach. Let $\{x_0, \dots, x_5\}$ be a set of homogeneous coordinates on $\mathbb{P}^5$ and assume that $P = \{x_0 = x_1 = x_2 = 0\}$. Then there exists quadrics Q_0, Q_1, Q_2 such that $X \subset \mathbb{P}^5$ is defined by a polynomial of the form

$$F(x_0, \dots, x_5) := x_0Q_0 + x_1Q_1 + x_2Q_2.$$

Let us express an arbitrary point in the base $\mathbb{P}^2$ as $[1 : a : b]$ without loss of generality. Then the preimage of $[1 : a : b]$ under f is determined by

$$F(1, a, b, x_3, x_4, x_5) = 0$$

which is a non-homogeneous quadric in x_3, x_4, x_5. Due to the generality of X, the Hessian of this quadric is a sextic non-homogeneous polynomial in a and b and defines a smooth sextic curve $C \subset \mathbb{P}^2$. Because a quadric degenerates if and only if its Hessian vanishes, this completes the proof. □

Let us assume that $X \in \mathcal{C}_8$ is general enough so that Lemma 4.9 holds and the degenerate fibers of $f'\colon \mathcal{Q} \to \mathbb{P}^2$ are at worst cones over conics (instead of unions of two planes). In this setting, the double cover over $\mathbb{P}^2$ branched over C is a K3 surface S. On the other hand, the relative Fano variety of lines

$$F(\mathcal{Q}/\mathbb{P}^2) = \{\text{lines contained in the fibers of } \mathcal{Q} \to \mathbb{P}^2\}$$

consists of two disjoint copies of $\mathbb{P}^1$ on a smooth fiber and a single $\mathbb{P}^1$ on a degenerate fiber. It follows that the base change

$$\begin{array}{ccc} \mathcal{M} := S \times_{\mathbb{P}^2} F(\mathcal{Q}/\mathbb{P}^2) & \longrightarrow & F(\mathcal{Q}/\mathbb{P}^2) \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathbb{P}^2 \end{array}$$

induces a $\mathbb{P}^1$-fibration $\mathcal{M} \to S$, which then corresponds to a class $\alpha \in \mathrm{Br}(S)$. Kuznetsov showed that there is an equivalence

$$\mathcal{A}_X \cong \mathcal{D}^b(S, \alpha). \tag{4.2}$$

This relation was built up by a semiorthogonal decomposition of $\mathcal{D}^b(\mathcal{Q})$ on one hand, the expression of $\mathcal{D}^b(S, \alpha)$ as certain derived category on $\mathbb{P}^2$ via the double cover $S \to \mathbb{P}^2$ on the other hand, and then compare these structures via the resolution diagram (4.1). We refer the reader to [Kuz10, Theorem 4.3] for the details.

Lemma 4.10.

(a) *There exists a map*

$$\{multisections\ of\ \mathcal{Q} \to \mathbb{P}^2\} \longrightarrow \{multisections\ of\ \mathcal{M} \to S\}$$

which preserves the degrees of multisections.

(b) *There exists a map*

$$\{multisections\ of\ \mathcal{M} \to S\} \longrightarrow \{multisections\ of\ \mathcal{Q} \to \mathbb{P}^2\}$$

which raises the degree of a multisection from d to d^2.

Proof. This is the main content of [Kuz10, Proposition 4.7]. To prove (a), let us start with a multisection $T \subset \mathcal{Q}$ of degree d. Then T meets each fiber in d points counted with multiplicity. If the fiber is smooth, then there are d lines in each of the two families of rulings incident to the d points. If the fiber is degenerate, then the same thing holds though there is only one family of rulings. These lines correspond to d points in each fiber of $\mathcal{M}$, thus T determines a multisection of $\mathcal{M} \to S$ of degree d.

Now we prove (b). Let $R \subset \mathcal{M}$ be a multisection of degree d. Pick a general point $p \in \mathbb{P}^2$, let $p', p'' \in S$ be its preimage points under the double cover $S \to \mathbb{P}^2$, and let $\mathcal{M}_{p'}$ and $\mathcal{M}_{p''}$ be their preimage fibers in the $\mathbb{P}^1$-fibration $\mathcal{M} \to S$. On the fiber quadric $\mathcal{Q}_p \subset \mathcal{Q}$ over p, the intersections $R \cap \mathcal{M}_{p'}$ and $R \cap \mathcal{M}_{p''}$ correspond respectively to d lines in each of the two families of rulings, and these $2d$ lines meet in totally d^2 points on $\mathcal{Q}_p$. Varying p gives a multisection of degree d^2 on $\mathcal{Q}$. □

Proposition 4.11. *The following statements are equivalent:*

(1) *The class α vanishes, i.e., $\mathcal{A}_X \cong \mathcal{D}^b(S)$ by (4.2).*

(2) *The $\mathbb{P}^1$-fibration $\mathcal{M} \to S$ admits a multisection of odd degree.*

(3) *The quadric fibration $\mathcal{Q} \to \mathbb{P}^2$ admits a multisection of odd degree.*

(4) *$X \in \mathcal{C}_\delta$, i.e., X contains a surface W such that $\delta(W) = \deg(W) - [W] \cdot [P]$ is odd.*

Proof. The equivalence (1) ⇔ (2) follows from the geometric interpretation of brauer classes. The equivalence (2) ⇔ (3) follows from Lemma 4.10. The equivalence (3) ⇔ (4) is a consequence of Lemma 4.7. □

4.3 Known examples of rational cubic fourfolds

One of the implications in Conjecture 4.4 is still wide open, namely, among the cubic fourfolds without an associated K3 surface, none of them is proved to be irrational. On the other hand, there are a few types of cubics with associated K3 surfaces are known to be rational. Before reviewing these examples, let us recall a result by Kontsevich and Tschinkel:

Theorem 4.12 ([KT19, Theorem 1])**.** *Let B be a smooth connected curve over a field of characteristic zero. Suppose that there are smooth proper morphisms*

$$\pi\colon \mathcal{X} \to B \quad \textit{and} \quad \pi'\colon \mathcal{X}' \to B$$

whose generic fibers are birational over the function field of B. Then, for every closed point $b \in B$, the fibers of π and π' over b are birational over the residue field at b. In particular, if the generic fiber of π is rational, then every fiber of π is rational.

Due to this theorem, to prove that the cubic fourfolds parametrized by an irreducible locus $\mathcal{D} \subset \mathcal{C}$ (e.g. $\mathcal{C}_d \subset \mathcal{C}$) are all rational, it is sufficient to prove that $\mathcal{D}$ contains a Zariski open subset whose members are rational. For example, the locus $\mathcal{C}_{\mathrm{Pf}} \subset \mathcal{C}_{14}$ of Pfaffian cubic fourfolds contains a subset which is Zariski open in $\mathcal{C}_{14}$, so the members of $\mathcal{C}_{14}$ are all rational because Pfaffian cubic fourfolds are rational.

In fact, the rationality of all members of $\mathcal{C}_{14}$ was first proved by Bolognesi, Russo, and Staglianò via degenerations of quartic scrolls [BRS19]. Besides $\mathcal{C}_{14}$, there are loci also known to parametrize rational cubic fourfolds:

- The divisors $\mathcal{C}_{26}$, $\mathcal{C}_{38}$, and $\mathcal{C}_{42}$ [RS19, RS20].
- Countably infinite collections of divisors in $\mathcal{C}_8$ [Has99] and $\mathcal{C}_{18}$ [AHTVA19].
- Some divisors in $\mathcal{C}_{20}$ [FL20].

This list is organized with respect to the methods for establishing the rationality. In the following, we will sketch these constructions case by case with respect to the list.

4.3.1 Congruences of conics 5-secant to a surface Computations with Macaulay2 [GS21] form an essential part in Russo and Staglianò's approach to the rationality. Their main idea is to verify that a general $X \in \mathcal{C}_d$, where $d = 26, 38, 42$, contains an irreducible surface $S = S_d$ which admits a special family $\mathcal{H}$ of curves which they call a *congruence of* $(3e-1)$*-secant curves of degree* e.

More precisely, a general $C \in \mathcal{H}$ is a smooth rational curve of degree e that intersects S in $3e-1$ points counted with multiplicity and, for a general $p \in \mathbb{P}^5$, there exists one and only one such curve passing through p. The space $\mathcal{H}$ and the universal family $\widetilde{\mathcal{H}}$ form a diagram

$$\begin{array}{ccc} \widetilde{\mathcal{H}} & & \\ {\scriptstyle\pi}\downarrow & \searrow^{\theta} & \\ \mathcal{H} & & \mathbb{P}^5 \end{array}$$

where θ is birational. Notice that $\mathcal{H}$ is irreducible and of dimension 4 in this setting. Suppose that X meets a general $C \in \mathcal{H}$ transversely. Then the intersection $X \cap C$ consists of $3e$ points with $3e-1$ of them lying on S, which realizes X as a rational section of $\pi\colon \widetilde{\mathcal{H}} \to \mathcal{H}$, hence X is birational to $\mathcal{H}$. According to [RS19, Theorem 1], if the rational map

$$\varphi\colon \mathbb{P}^5 \dashrightarrow |\mathcal{I}_S(3)| \cong \mathbb{P}^n.$$

is birational onto its image, then the members of $|\mathcal{I}_S(3)|$ with at worst rational singularity realize as sections as above. In particular, they are rational provided that $\mathcal{H}$ is rational.

Notice that a cubic hypersurface $Y \subset \mathbb{P}^5$ with an ordinary double point $q \in Y$ is rational as the projection from q defines a birational map $Y \dashrightarrow \mathbb{P}^4$. Therefore, to prove that $\mathcal{H}$ is rational, it is sufficient to find one member of $|\mathcal{I}_S(3)|$ with an ordinary double point. Another method to establish the rationality of $\mathcal{H}$ is via the *trisecant flops* introduced in [RS20, §2]. The benefit of the second approach is that it is able to reveal the birational incarnations of the associated K3 surfaces in the rational parametrizations.

Remark 4.13. In the case $d = 26$, the surface $S_{26} \subset \mathbb{P}^5$ is a septic surface with a node, constructed as the projection of a septic del Pezzo surface $\Sigma \subset \mathbb{P}^7$ from a line which meets the secant variety $\mathrm{Sec}(\Sigma)$ transversely at one point. In the case $d = 38$, the surface $S_{38} \subset \mathbb{P}^5$ is a smooth surface of degree 10 and sectional genus 6, obtained as the image of $\mathbb{P}^2$ via the linear system of curves of degree 10 with 10 fixed triple points. In these two cases, the congruences consist of 5-secant conics. For $d = 42$, the construction of S_{42} requires a lot more space to describe, so we refer the reader to the original paper [RS20, §4.1].

4.3.2 Fibrations in rational surfaces over the plane As exhibited in §4.2.1, every cubic fourfold $X \in \mathcal{C}_8$ contains a plane P, and the projection from P induces a quadric fibration

$$\mathcal{Q} := \mathrm{Bl}_P X \longrightarrow \mathbb{P}^2$$

which is rational if and only if it admits a multisection of odd degree. Moreover, the last condition characterizes a countably infinite collection of divisors in $\mathcal{C}_8$.

The situation for $\mathcal{C}_{18}$ is similar. In this case, a general $X \in \mathcal{C}_{18}$ contains an elliptic ruled surface $T \subset X$, that is, a fibration in lines over an elliptic curve, such that the system of quadrics passing through T defines a rational map

$$\Pi \colon \mathbb{P}^5 \dashrightarrow \mathbb{P}^2$$

whose base locus is T union with two planes [AHTVA19, Theorem 2]. The surface T is cut out by cubics in $\mathbb{P}^5$ [AHTVA19, Proposition 4], so the restriction of Π to X defines a rational map with exactly T as the base locus. Resolving this map gives the diagram

$$\begin{array}{ccc} \mathcal{S} := \mathrm{Bl}_T X & & \\ \downarrow & \overset{\pi'}{\searrow} & \\ X & \underset{\pi}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

where the generic fiber of π' is a del Pezzo surface of degree 6 [AHTVA19, Theorem 6]. Let us remark that, if $S \subset X$ is the image of a general fiber of π', then the classes h^2 and $[S]$ span a sublattice in $H^4(X, \mathbb{Z})$ with intersection matrix

$$\begin{pmatrix} 3 & 6 \\ 6 & 18 \end{pmatrix} \tag{4.3}$$

which is a labelling of discriminant 18. Similar to quadric surfaces, we have the following property about the rationality of sextic del Pezzo surfaces:

Proposition 4.14 ([AHTVA19, Proposition 8]). *Let $\mathcal{S}$ be a sextic del Pezzo surface over a perfect field k. Then the following statements are equivalent:*

- *$\mathcal{S}$ is rational over k.*
- *$\mathcal{S}$ contains a k-rational point.*
- *$\mathcal{S}$ contains a k'-rational point for some extension k'/k of degree prime to 6.*

By Proposition 4.14, the cubic fourfold X is rational if there exists a surface $\Sigma \subset X$ which lifts to a section of π'. An analysis on lattices shows that such cubics form a codimension one locus in $\mathcal{C}_{18}$. More precisely, $[\Sigma] \in H^4(X, \mathbb{Z})$ and the labelling (4.3) span a sublattice

$$\begin{pmatrix} 3 & 6 & a \\ 6 & 18 & 1 \\ a & 1 & b \end{pmatrix} \text{ of discriminant } \Delta = -3 + 12a - 18a^2 + 18b. \tag{4.4}$$

We can set $a = -1, 0, 1$ upon replacing $[\Sigma]$ with $[\Sigma] + m(2h^2 - [S])$ for a suitable $m \in \mathbb{Z}$, thus positive integers $\Delta \equiv 9 \pmod{12}$ arise as discriminants, each for precisely one lattice. For every such Δ, let $\mathcal{C}_\Delta \subset \mathcal{C}_{18}$ be the divisor of X labelled by (4.4). Then all but finitely many $\mathcal{C}_\Delta$ is nonempty, and the union $\bigcup \mathcal{C}_\Delta \subset \mathcal{C}_{18}$ gives the desired locus. We refer the reader to [AHTVA19, §4] for the details about this part.

4.3.3 Birational maps defined by the Veronese surface Let $X \subset \mathbb{P}^5$ be a cubic fourfold containing a Veronese surface V, namely, the embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$ via the linear system of conics. Then the classes h^2 and $[V]$ span the labelling

$$\begin{pmatrix} 3 & 4 \\ 4 & 12 \end{pmatrix} \subset H^4(X, \mathbb{Z})$$

of discriminant 20, which shows that $X \in \mathcal{C}_{20}$. On the other hand, it can be proved that every member of $\mathcal{C}_{20} \setminus \mathcal{C}_8$ contains V [FL20, Proposition 2.1]. Notice that the rational cubic fourfolds introduced previously all belong to $\mathcal{C}_d$ with d in the list

$$\{8,\ 14,\ 18,\ 26,\ 38,\ 42\}. \tag{4.5}$$

The following result gives loci of rational cubic fourfolds in $\mathcal{C}_{20}$ outside these divisors.

Theorem 4.15 ([FL20, Theorem 1.3]). *There exists a birational involution σ on $\mathcal{C}_{20}$ such that, for each $d = 26, 38, 42$, it maps a component of the intersection $\mathcal{C}_{20} \cap \mathcal{C}_d$ birationally onto a divisor $\mathcal{D} \subset \mathcal{C}_{20}$ not contained in $\mathcal{C}_{d'}$ for all d' in (4.5). In particular, there exist at least three irreducible divisors in $\mathcal{C}_{20}$ consisting of cubic fourfolds whose rationality is not known before.*

The birational involution σ is constructed from the surface V. More precisely, the linear system of quadrics cutting out V has dimension 5, thus defines a rational map

$$F\colon \mathbb{P}^5 \overset{\simeq}{\dashrightarrow} \mathbb{P}^5 \tag{4.6}$$

which turns out to be birational (see Remark 4.16). Up to a change of coordinates, we may assume that it is involutive, i.e., $F = F^{-1}$, and is defined by the 2-minors of the symmetric matrix

$$\begin{pmatrix} x_0 & x_1 & x_5 \\ x_1 & x_2 & x_3 \\ x_5 & x_3 & x_4 \end{pmatrix}. \tag{4.7}$$

The exceptional divisor of F coincides with the secant variety of V, where the latter is defined by the determinant of (4.7), thus is a cubic hypersurface singular along V. Therefore, the restriction of F to a smooth cubic $X \subset \mathbb{P}^5$ containing V induces a birational map

$$f \colon X \overset{\simeq}{\dashrightarrow} X' \quad \text{where} \quad X' = F(X) \subset \mathbb{P}^5.$$

Notice that this can be resolved by a single blowup as below

$$\begin{array}{ccc} & Y := \mathrm{Bl}_V X & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\pi'} \\ X & \underset{f}{\overset{\sim}{\dashrightarrow}} & X'. \end{array} \tag{4.8}$$

Remark 4.16. A birational automorphism of $\mathbb{P}^n$ is called a *Cremona transformation*. The map F is one of the only seven types of Cremona transformations with an irreducible smooth curve or surface as base locus [CK89, Theorems 2.2 & 3.3]. If one allows surfaces with at worst non-normal double points, then there is only one additional possibility, which are Cremona transformations of $\mathbb{P}^4$ with base loci birational to K3 surfaces of degree 12. We refer the reader to [HL18] for the details.

Lemma 4.17. *The variety $X' \subset \mathbb{P}^5$ is a cubic hypersurface containing V.*

Proof. One way to prove the statement is to resolve F as

$$\begin{array}{ccc} & \Gamma := \mathrm{Bl}_V \mathbb{P}^5 & \\ {\scriptstyle\pi_1}\swarrow & & \searrow{\scriptstyle\pi_2} \\ \mathbb{P}^5 & \underset{F}{\overset{\sim}{\dashrightarrow}} & \mathbb{P}^5 \end{array}$$

and then to compute the strict transform of X' to Γ in the Picard group in terms of the pullback of the hyperplane class under π_2 and the exceptional divisor of π_1. We refer the reader to [FL20, Proposition 3.2] for the details. Here we provide an elementary approach. Suppose that F is expressed explicitly as

$$y_i = Q_i(x_0, \ldots, x_5), \quad i = 0, \ldots, 5, \tag{4.9}$$

where $Q_0, \ldots, Q_5$ are quadrics that cut V out in $\mathbb{P}^5$. Then X is defined by a polynomial of the form $\sum_{i=0}^{5} L_i Q_i$ where $L_0, \ldots, L_5$ are linear in $x_0, \cdots, x_5$. Since $F = F^{-1}$, we can express F^{-1} in the same way as (4.9) with x_i replaced by y_i and vice versa. Therefore, X' is defined by the polynomial $\sum_{i=0}^{5} L_i y_i$ where $L_0, \ldots, L_5$ are linear in $Q_0(y), \ldots, Q_5(y)$. In particular, X' is defined by a cubic equation in $y_0, \ldots, y_5$ vanishing along V. □

From Lemma 4.17, we get a birational involution

$$\sigma\colon \mathcal{C}_{20} \overset{\simeq}{\dashrightarrow} \mathcal{C}_{20} : X \longmapsto F(X) \tag{4.10}$$

and we would like to know how it acts on the intersection $\mathcal{C}_{20} \cap \mathcal{C}_d$, especially for d in the list 4.5. An intersection-theoretic computation on the resolution (4.8) shows that a very general $X \in \mathcal{C}_{20} \cap \mathcal{C}_d$ is taken to $X' \in \mathcal{C}_{20}$ such that there is a transformation law

$$H^{2,2}(X,\mathbb{Z}) \cong \begin{pmatrix} 3 & 4 & a \\ 4 & 12 & b \\ a & b & c \end{pmatrix} \longmapsto \begin{pmatrix} 3 & 4 & 4a-b \\ 4 & 12 & b \\ 4a-b & b & c+(3a-b)^2 \end{pmatrix} \cong H^{2,2}(X',\mathbb{Z}).$$

Theorem 4.15 was then proved by analyzing which labellings can be contained in the right hand side. As an explicit example, one can find a component $\mathcal{D} \subset \mathcal{C}_{20} \cap \mathcal{C}_{26}$ on which the transformation law has the form

$$\begin{pmatrix} 3 & 4 & 1 \\ 4 & 12 & 1 \\ 1 & 1 & 9 \end{pmatrix} \longmapsto \begin{pmatrix} 3 & 4 & 3 \\ 4 & 12 & 1 \\ 3 & 1 & 13 \end{pmatrix}.$$

Then a computation shows that the lattice on the right bears a labelling of discriminant 146 but no labelling of discriminant listed in (4.5) [FL20, Theorem 3.13]. (The discriminants 2 and 6 also need to be ruled out in order to confirm that the image cubics are smooth.) Notice that 146 satisfies the criterion for the existence of an associated K3 surface (Theorem 3.4). As a consequence, the image component $\sigma(\mathcal{D}) \subset \mathcal{C}_{20} \cap \mathcal{C}_{146}$ parametrizes cubic fourfolds whose rationality is not known before.

Remark 4.18. Two cubic fourfolds X and X' are called *Fourier–Mukai partners* if their K3 categories $\mathcal{A}_X$ and $\mathcal{A}_{X'}$ are equivalent. The involution (4.10) takes a very general member of $\mathcal{C}_{20}$ to its unique non-isomorphic Fourier–Mukai partner [FL20, Theorem 1.1].

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