ERRATUM TO "TRANSVERSE LINES TO SURFACES OVER FINITE FIELDS"

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In [ADL21, Theorem 4.5], we claim that, given a reduced and geometrically irreducible hypersurface $X = \{F = 0\} \subset \mathbb{P}^n$ over $\mathbb{F}_q$, if it is Frobenius nonclassical, then it is nonreflexive. If we denote by $\gamma\colon X \dashrightarrow X^*$ the Gauss map of X, then X is nonreflexive if and only if, at a general point $P \in X$, the pullback of differentials

$$d\gamma_P^*\colon (\gamma^*\Omega_{X^*}) \otimes \mathbb{F}_q(P) \longrightarrow \Omega_X \otimes \mathbb{F}_q(P) \tag{0.1}$$

is not injective. In our original argument, we intended to prove this property via [ADL21, Lemma 4.6], which shows that the determinant of the Hessian matrix

$$H_F := (F_{ij}) \quad \text{where} \quad F_{ij} = \frac{\partial^2 F}{\partial X_j \partial X_i}$$

is zero modulo F. While [ADL21, Lemma 4.6] is correct, this is not sufficient to prove [ADL21, Theorem 4.5]. The subtle error is the following *incorrect* assertion: a hypersurface X is nonreflexive if and only if the determinant of the Hessian matrix of X vanishes identically on X. Let us explain why this claim fails when $\deg(X) \equiv 1 \mod \operatorname{char}(\mathbb{F}_q)$.

Indeed, if we denote $d = \deg(F)$ and $F_i = \partial F/\partial X_i$, then a computation with Euler's formula shows

$$\begin{pmatrix} d(d-1)F & (d-1)F_1 & \cdots & (d-1)F_n \\ (d-1)F_1 & F_{11} & \cdots & F_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ (d-1)F_n & F_{1n} & \cdots & F_{nn} \end{pmatrix} = \begin{pmatrix} X_0 & X_1 & \cdots & X_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \cdot H_F \cdot \begin{pmatrix} X_0 & 0 & \cdots & 0 \\ X_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ X_n & 0 & \cdots & 1 \end{pmatrix} \tag{0.2}$$

from which we see that $\det(H_F)$ is constantly zero whenever $d \equiv 1 \mod \operatorname{char}(\mathbb{F}_q)$. Moreover, we actually assume that the Gauss map is finite in our original argument, but we mistakenly ignored this assumption in the statement of [ADL21, Theorem 4.5]. In this corrigendum, we fix the statement and prove it without using [ADL21, Lemma 4.6].

Theorem 0.1. *Let $X \subset \mathbb{P}^n$ be a reduced and geometrically irreducible Frobenius nonclassical hypersurface over $\mathbb{F}_q$. Suppose that $\dim(X) = \dim(X^*)$ (which is satisfied when X is smooth due to Zak's theorem [Zak93]). Then X is nonreflexive.*

Our strategy of proof goes as follows: Let I and I' be the ideal sheaves for X and X^*, respectively. Then there is a commutative diagram for sheaves of differentials:

$$\begin{tikzcd} \gamma^*\left(I'/I'^2\right) \arrow[r] \arrow[d] & \gamma^*\left(\Omega_{(\mathbb{P}^n)^*}|_{X^*}\right) \arrow[r] \arrow[d] & \gamma^*\Omega_{X^*} \arrow[r] \arrow[d, "d\gamma^*"] & 0 \\ I/I^2 \arrow[r] & \Omega_{\mathbb{P}^n}|_X \arrow[r] & \Omega_X \arrow[r] & 0. \end{tikzcd} \tag{0.3}$$

Note that $\gamma = \Gamma|_X$ where Γ is the polar map

$$\Gamma\colon \mathbb{P}^n \dashrightarrow (\mathbb{P}^n)^* : [X_0 : \cdots : X_n] \longmapsto \left[\frac{\partial F}{\partial X_0} : \cdots : \frac{\partial F}{\partial X_n}\right].$$

We have $\gamma^*\left(\Omega_{(\mathbb{P}^n)^*}|_{X^*}\right) = \left(\Gamma^*\Omega_{(\mathbb{P}^n)^*}\right)|_X$ and the vertical arrow in the middle of (0.3) is induced by the pullback of differentials $d\Gamma^*\colon \Gamma^*\Omega_{(\mathbb{P}^n)^*} \longrightarrow \Omega_{\mathbb{P}^n}$. Let $U \subset \mathbb{P}^n$ be an open neighborhood of P where $I(U) = (f)$. In order to prove that (0.1) is not injective, we will prove that the image of the linear map

$$d\Gamma^*_P\colon (\Gamma^*\Omega_{(\mathbb{P}^n)^*}) \otimes \mathbb{F}_q(P) \longrightarrow \Omega_{\mathbb{P}^n} \otimes \mathbb{F}_q(P) \tag{0.4}$$

has dimension at most $n-2 = \dim(X) - 1$ modulo df.

1. Proof of the theorem

Let $[Y_0 : \cdots : Y_n]$ be homogeneous coordinates for $(\mathbb{P}^n)^*$ so that the polar map Γ can be written as $Y_i = F_i$, and let $y_i = Y_i/Y_0$ be the affine coordinates for the chart $\{Y_0 \neq 0\}$. Assume without loss of generality that the point $P \in X$ belongs to the open subset

$$U := \{X_0 \neq 0\} \cap \{F_0 \neq 0\} \cap \left(\bigcup_{i=1}^{n}\{F_i \neq 0\}\right) \subset \mathbb{P}^n.$$

If we write $x_i := X_i/X_0$ and let $f_i = f_i(x_1, \ldots, x_n)$ be the dehomogenization of F_i with respect to X_0, then $\Gamma|_U$ can be expressed as $y_i = f_i/f_0 = F_i/F_0$. In this setting, the map of differentials $d\Gamma^*|_U$ sends each dy_i to

$$dy_i = \sum_{j=1}^{n} \frac{\partial(f_i/f_0)}{\partial x_j} dx_j = \sum_{j=1}^{n} \left(\frac{(\partial f_i/\partial x_j) f_0 - f_i(\partial f_0/\partial x_j)}{f_0^2}\right) dx_j = \sum_{j=1}^{n} \left(\frac{F_{ij}F_0 - F_iF_{0j}}{F_0^2}\right) dx_j$$

This linear map corresponds to the square matrix M_F/F_0 where M_F is given by

$$M_F := \begin{pmatrix} F_{11} - \frac{F_1}{F_0}F_{01} & F_{21} - \frac{F_2}{F_0}F_{01} & \cdots & F_{n1} - \frac{F_n}{F_0}F_{01} \\ F_{12} - \frac{F_1}{F_0}F_{02} & F_{22} - \frac{F_2}{F_0}F_{02} & \cdots & F_{n2} - \frac{F_n}{F_0}F_{02} \\ \vdots & \vdots & \ddots & \vdots \\ F_{1n} - \frac{F_1}{F_0}F_{0n} & F_{2n} - \frac{F_2}{F_0}F_{0n} & \cdots & F_{nn} - \frac{F_n}{F_0}F_{0n} \end{pmatrix}.$$

Now we extend the above matrix to the following one

$$H''_F := \begin{pmatrix} 0 & F_1 & F_2 & \cdots & F_n \\ F_1 & F_{11} - \frac{F_1}{F_0}F_{01} & F_{21} - \frac{F_2}{F_0}F_{01} & \cdots & F_{n1} - \frac{F_n}{F_0}F_{01} \\ F_2 & F_{12} - \frac{F_1}{F_0}F_{02} & F_{22} - \frac{F_2}{F_0}F_{02} & \cdots & F_{n2} - \frac{F_n}{F_0}F_{02} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_n & F_{1n} - \frac{F_1}{F_0}F_{0n} & F_{2n} - \frac{F_2}{F_0}F_{0n} & \cdots & F_{nn} - \frac{F_n}{F_0}F_{0n} \end{pmatrix}.$$

Lemma 1.1. *Let $(0, a_1, \ldots, a_n)$ be a nonzero vector where $a_i \in \overline{\mathbb{F}_q}$. If*

$$(0, a_1, \ldots, a_n) \cdot H''_F(P) = 0 \qquad \text{where} \qquad P \in U,$$

then the column space of $M_F(P)$ has dimension $\leq n-2$ modulo $(F_1(P), \ldots, F_n(P))^t$.

Proof. The hypothesis implies that

$$(a_1,\dots,a_n)\cdot M_F(P)=0 \qquad \text{and} \qquad (a_1,\dots,a_n)\cdot (F_1(P),\dots,F_n(P))^t=0.$$

The first equation implies that of $\operatorname{rk}(M_F(P))\le n-1$. If $\operatorname{rk}(M_F(P))\le n-2$, the proof is done. If $\operatorname{rk}(M_F(P))=n-1$, the second equation above implies that $(F_1(P),\dots,F_n(P))^t$ belongs to the column space of $M_F(P)$, which proves the claim. $\square$

The matrix H''_F is related to the Hessian matrix H_F in the following way: If we restrict the matrix on the left hand side of (0.2) to $X=\{F=0\}$ and then divide its first row and first column by $(d-1)$, we will get

$$H'_F := \begin{pmatrix} 0 & F_1 & \cdots & F_n \\ F_1 & F_{11} & \cdots & F_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ F_n & F_{1n} & \cdots & F_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{F_{01}}{F_0} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{F_{0n}}{F_0} & 0 & \cdots & 1 \end{pmatrix} \cdot H''_F. \tag{1.1}$$

However, we cannot divide by $d-1$ in the field when $d\equiv 1 \mod p$. We see that the matrix H''_F is the correct replacement for the usual Hessian in positive characteristic, especially when $d\equiv 1 \mod p$.

On the other hand, X is Frobenius nonclassical means that there exists a polynomial R that satisfies

$$FR = X_0^qF_0 + X_1^qF_1 + \cdots + X_n^qF_n. \tag{1.2}$$

Lemma 1.2. *Let $P=[1:X_1:\cdots:X_n]\in X\cap U$. Then*

$$(1-d+R, X_1-X_1^q, X_2-X_2^q,\cdots, X_n-X_n^q)\cdot H'_F(P)=0.$$

Proof. Subtracting Euler's formula $dF=X_0F_0+\cdots+X_nF_n$ by (1.2) gives

$$(d-R)F=(X_0-X_0^q)F_0+(X_1-X_1^q)F_1+\cdots+(X_n-X_n^q)F_n. \tag{1.3}$$

Taking partial derivatives of both sides with respect to X_i followed by a rearrangement gives

$$-R_iF=(1-d+R)F_i+(X_0-X_0^q)F_{0i}+(X_1-X_1^q)F_{1i}+\cdots+(X_n-X_n^q)F_{ni} \tag{1.4}$$

Then the statement follows by a straightforward computation with (1.3), (1.4), the hypothesis that $X_0=1$, and the fact that $F(P)=0$. $\square$

Proof of Theorem 0.1. Pick a general $P=[1:X_1:\cdots:X_n]\in X\cap U$. First we compute

$$\begin{aligned}
&(1-d+R, X_1-X_1^q, X_2-X_2^q,\cdots, X_n-X_n^q)\cdot \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \frac{F_{01}}{F_0} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{F_{0n}}{F_0} & 0 & \cdots & 1 \end{pmatrix}\\
&= \left((1-d+R)+\sum_{j=1}^n\left(\frac{F_{0j}}{F_0}(X_j-X_j^q)\right), X_1-X_1^q,\dots,X_n-X_n^q\right)\\
&\overset{(1.4)}{=} (0, X_1-X_1^q,\dots,X_n-X_n^q)
\end{aligned}$$

Lemma 1.2 and relation (1.1) implies that $(0,X_1-X_1^q,\dots,X_n-X_n^q)\cdot H''_F(P)=0$, which implies that $(X_1-X_1^q,\dots,X_n-X_n^q)\cdot M_F(P)=0$. By applying Lemma 1.1, we conclude that the image of $d\gamma_P^*$ has dimension $\leq n-2$ modulo $\sum_{i=1}^n F_i(P)dx_i = df$, thus it cannot be injective. This shows that the Gauss map γ is inseparable, whence X is nonreflexive. $\square$

References

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[Zak93] F. L. Zak, *Tangents and secants of algebraic varieties*, Translations of Mathematical Monographs, vol. 127, American Mathematical Society, Providence, RI, 1993. Translated from the Russian manuscript by the author.