Nielsen realization problem for derived automorphisms of generic K3 surfaces

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Abstract

We prove that all nontrivial finite subgroups of derived automorphisms of K3 surfaces of Picard number one have order two and give formulas for the numbers of their conjugacy classes. We also obtain a similar result for the subgroups which are finite up to shifts. This in turn shows that such a K3 surface admits an associated cubic fourfold if and only if it has a derived automorphism of order three up to shifts. These results are achieved by proving that such a subgroup fixes a Bridgeland stability condition up to $\mathbb{C}$-actions. We also establish similar existence results for curves, twisted abelian surfaces, generic twisted K3 surfaces, and standard autoequivalences on surfaces.

Contents

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1 Introduction

For a K3 surface X of Picard number one and degree $2n$, it is well-known that

$$\mathrm{Aut}(X) = \begin{cases} \{\mathrm{id}\} & \text{if} \quad n \geq 2 \\ \langle \iota \rangle \cong \mathbb{Z}_2 & \text{if} \quad n = 1 \end{cases}$$

where ι is the covering involution of $X \to \mathbb{P}^2$ [Nik81, Corollary 10.1.3]. Now consider the bounded derived category of coherent sheaves $\mathrm{D^b}(X)$. Then $\mathrm{Aut}(X)$ appears as a subgroup of the group of autoequivalences $\mathrm{Aut}(\mathrm{D^b}(X))$. Does the latter contains a nontrivial element of finite order other than ι? For quartic K3 surfaces, that is, when $n = 2$, one can consider the autoequivalence

$$\Theta \coloneqq (- \otimes \mathcal{O}_X(1)) \circ T_{\mathcal{O}_X}$$

where $T_{\mathcal{O}_X}$ is the spherical twist along $\mathcal{O}_X$. It holds that $\Theta^4 = [2]$ ([CK08],[BFK12, Proposition 5.8]), so the autoequivalence $\Theta^2[-1]$ gives an involution. In general, it is a nontrivial task to construct finite order autoequivalences.

As part of the main results in this paper, we give a complete classification of finite subgroups of autoequivalences as follows:

Theorem 1.1. *Let X be a K3 surface of Picard number one and degree $2n$. Then every nontrivial finite subgroup of $\mathrm{Aut}(\mathrm{D^b}(X))$ is of order 2 and generated by an anti-symplectic involution. Moreover, the number of conjugacy classes of these subgroups is equal to*

$$\begin{cases} 1 & \text{if} \quad n = 1, 2 \\ 2^{a-1} & \text{if} \quad n = 2^\ell p_1^{k_1} \cdots p_a^{k_a}, \quad \ell \in \{0, 1\}, \quad a, k_i \geq 1, \quad p_i \equiv 1 \ (\mathrm{mod}\ 4) \text{ are primes} \\ 0 & \text{if} \quad n \text{ is divisible by } 4 \text{ or a prime } p \equiv 3 \ (\mathrm{mod}\ 4). \end{cases}$$

It is also interesting to classify subgroups of autoequivalences which are finite modulo shift functors. For instance, if a K3 surface X admits an associated cubic fourfold $Y \subseteq \mathbb{P}^5$ so that $\mathrm{D^b}(X)$ is equivalent to the Kuznetsov component of $\mathrm{D^b}(Y)$, then there exists an element of order 3 in $\mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2]$ by [Kuz04, Lemma 4.2]. In [Huy23, Remark 7.1], Huybrechts asks whether the converse of this statement is true or not at both categorical and cohomological levels. In this paper, we provide a complete classification and counting formulas for finite subgroups of $\mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2]$. Based on this observation, we give an affirmative answer to Huybrechts' question for K3 surfaces of Picard number one.

Theorem 1.2. *Let X be a K3 surface of Picard number one and degree $2n$. Then every maximal finite subgroup of $G \subseteq \mathrm{Aut}(\mathrm{D^b}(X))/\mathbb{Z}[2]$ is isomorphic to $G_s \times \mathbb{Z}_2[1]$, where $G_s \subseteq G$ is the symplectic subgroup of G and has the form*

$$G_s \cong \begin{cases} \mathbb{Z}_6 & \text{if} \quad n = 1 \\ \mathbb{Z}_4 & \text{if} \quad n = 2 \\ 0,\ \mathbb{Z}_2,\ \text{or}\ \mathbb{Z}_3 & \text{if} \quad n \geq 3. \end{cases}$$

For $n = 1, 2$, there exists one and only one such subgroup up to conjugation. For $n \geq 3$, the number of subgroups isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2[1]$ up to conjugation is the same as the number

of finite subgroups of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ *up to conjugation. On the other hand, the number of conjugacy classes of subgroups isomorphic to* $\mathbb{Z}_3 \times \mathbb{Z}_2[1]$ *is equal to*

$$\begin{cases} 2^{a-1} & \text{if} \quad n = 3^{\ell} p_1^{k_1} \cdots p_a^{k_a}, \quad \ell \in \{0,1\}, \quad a, k_i \geq 1, \quad p_i \equiv 1 \ (\mathit{mod}\ 3) \ \text{are primes} \\ 0 & \text{if} \quad n \ \text{is divisible by } 9 \ \text{or a prime } p \equiv 2 \ (\mathit{mod}\ 3). \end{cases}$$

For K3 surfaces of Picard number one and degree at least 4, the following proposition provides a criterion for the existence of an associated cubic fourfold. We refer the reader to Proposition 6.1 for a complete list of equivalent criteria and Remark 6.2 for a discussion on the degree 2 case.

Proposition 1.3. *Let* X *be a K3 surface of Picard number one and degree at least* 4*. Then* X *admits an associated cubic fourfold if and only if* $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ *contains an element of order* 3*.*

Theorems 1.1 and 1.2 are achieved along our way in studying a categorical analogue of the *Nielsen realization problem*, which is the starting point of this paper. In the original version of the realization problem, one considers an oriented surface S and asks whether every finite subgroup of the mapping class group $\mathrm{Mod}(S)$ fixes a point on the Teichmüller space $\mathrm{Teich}(S)$ or not.[1] This problem was posted by Nielsen [Nie32] and answered positively by Kerckhoff [Ker83]. In light of various connections established in the last decade between Teichmüller theory and the theory of stability conditions on triangulated categories [GMN13, DHKK14, BS15, HKK17], one can formulate the following questions:

Let $\mathscr{D}$ be a triangulated category and $\mathrm{Stab}(\mathscr{D})$ be the space of stability conditions on $\mathscr{D}$ introduced by Bridgeland [Bri07].

Does every finite subgroup $G \subseteq \mathrm{Aut}(\mathscr{D})$ *fix a point on* $\mathrm{Stab}(\mathscr{D})$*?*

The space $\mathrm{Stab}(\mathscr{D})$ carries a free $\mathbb{C}$-action such that the double shift functor [2] acts trivially on the quotient manifold $\mathrm{Stab}(\mathscr{D})/\mathbb{C}$. Because shift functors have no classical counterpart in Teichmüller theory, it is also natural to consider the modified question:

Does every finite subgroup $G \subseteq \mathrm{Aut}(\mathscr{D})/\mathbb{Z}[2]$ *fix a point on* $\mathrm{Stab}(\mathscr{D})/\mathbb{C}$*?*

In fact, an affirmative answer to the second question implies an affirmative answer to the first one; see Lemma 2.3.

In this paper, we give affirmative answers to these questions when $\mathscr{D}$ is the bounded derived category of coherent sheaves on a curve, a twisted abelian surface, or a generic twisted K3 surface; see Propositions 2.4 and 3.7. For surfaces in general, we prove that these questions have positive answers if one considers only standard autoequivalences; see Proposition 2.10. For the main objects of study in this paper, namely, K3 surfaces of Picard number one, we obtain a more precise statement:

Theorem 1.4. *Let* X *be a projective K3 surface,* $\mathrm{Stab}^{\dagger}(X) \subseteq \mathrm{Stab}(X)$ *be the connected component which contains geometric stability conditions, and* $G \subseteq \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ *be a subgroup. If* X *has Picard number one, then*

$$G \text{ is finite} \iff G \text{ fixes a point on } \mathrm{Stab}^{\dagger}(X)/\mathbb{C}.$$

In general, the implication "$\Longleftarrow$*" holds regardless of the Picard number of* X*.*

[1] Farb–Looijenga [FL21] recently studied various versions of the realization problem for K3 manifolds.

To gain the intuition about how Theorem 1.4 imposes restrictions on autoequivalences of finite orders and thus helps us classify finite subgroups of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$, we refer the reader to Lemma 5.1. In the following, we briefly introduce our approach to the realization problem for K3 surfaces of Picard number one. The cases of curves, twisted abelian surfaces, and generic twisted K3 surfaces are simplifications of the same strategy:

For every K3 surface X, there is an $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$-equivariant covering map

$$\mathcal{S} := \mathrm{Stab}^{\dagger}(X)/\widetilde{\mathrm{GL}}^{+}(2,\mathbb{R}) \longrightarrow \mathcal{Q}_0^{+}(X) \tag{1.1}$$

where the group $\widetilde{\mathrm{GL}}^{+}(2,\mathbb{R})$ is the universal cover of $\mathrm{GL}^{+}(2,\mathbb{R})$ [Bri08, Theorem 1.1]. In the case that X has Picard number one, $\mathcal{Q}_0^{+}(X)$ is isomorphic to the upper half plane

$$\mathcal{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

with a discrete subset of points removed. Based on works by Bayer–Bridgeland [BB17] and Kawatani [Kaw14, Kaw19], we prove that every *symplectic* finite subgroup

$$G_s \subseteq \mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$$

has a fixed point on $\mathrm{Stab}^{\dagger}(X)/\mathbb{C}$ via the following two steps:

- If X has degree $2n$, then the actions of autoequivalences on $\mathcal{Q}_0^{+}(X) \subseteq \mathcal{H}$ form a subgroup $\Gamma_0^{+}(n) \subseteq \mathrm{PSL}(2,\mathbb{C})$ called the *Fricke group* [Kaw14, Proposition 2.9]. The space $\mathrm{Stab}^{\dagger}(X)$ is simply-connected [BB17, Theorem 1.3], so we get an isomorphism

$$\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2] \cong \pi_1^{\mathrm{orb}}(\Gamma_0^{+}(n) \setminus\!\setminus \mathcal{Q}_0^{+}(X)) \tag{1.2}$$

  (which requires some modification when $n = 1$; see Section 4.3). This shows that G_s is isomorphic to a cyclic group fixing a point $p \in \mathcal{Q}_0^{+}(X)$. Thus it fixes a point on $\mathcal{S}$ over p upon composing a deck transformation of (1.1).

- As $\mathrm{Stab}^{\dagger}(X)$ is simply connected, (1.1) is a universal cover. This property and [Kaw19, Theorem 1.3] assert that the group of deck transformations of (1.1) is freely generated by the squares of spherical twists [BB17, Theorem 4.1]. The freeness allows us to prove that G_s fixes some $\overline{\sigma} \in \mathcal{S}$ over p *without* composing a deck transformation. The finiteness of G_s then implies that it fixes a lift of $\overline{\sigma}$ in $\mathrm{Stab}^{\dagger}(X)/\mathbb{C}$.

To extend this result to any finite subgroup $G \subseteq \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$, we first prove that G is either symplectic or isomorphic to $G_s \times \mathbb{Z}_2[1]$ where $G_s \subseteq G$ is the symplectic subgroup; see Lemma 4.12. Since G_s has a fixed point, G has a fixed point as well. We remark that Theorem 1.2 is proved by computing the orbifold fundamental group in (1.2).

A stability condition representing a fixed point on $\mathrm{Stab}(\mathscr{D})/\mathbb{C}$ under the action of a non-shift functor is called a *Gepner type stability condition*. This notion was introduced by Toda in his study of Gepner points of stringy Kähler moduli spaces of Calabi–Yau manifolds [Tod14]. In view of this and the realization problem, it is reasonable to introduce the notion of *Gepner type autoequivalences* as follows:

Definition 1.5. Let $\mathscr{D}$ be a triangulated category. We say an autoequivalence $\Phi \in \mathrm{Aut}(\mathscr{D})$ is of *Gepner type* if its action on $\mathrm{Stab}(\mathscr{D})/\mathbb{C}$ has a fixed point, or equivalently, if there exists a stability condition σ such that

$$\Phi(\sigma) = \sigma \cdot \lambda \qquad \text{for some} \qquad \lambda \in \mathbb{C}.$$

More generally, we say a subgroup $G \subseteq \mathrm{Aut}(\mathscr{D})/\mathbb{Z}[m]$, where $m \in \mathbb{Z}$, is of Gepner type if its action on $\mathrm{Stab}(\mathscr{D})/\mathbb{C}$ has a fixed point. Autoequivalences of Gepner type with respect to a stability condition σ form a group which we denote as

$$\mathrm{Aut}(\mathscr{D}, \sigma \cdot \mathbb{C}) = \{\Phi \in \mathrm{Aut}(\mathscr{D}) \mid \Phi(\sigma) \in \sigma \cdot \mathbb{C}\}.$$

Its subgroup of elements fixing σ is denoted as $\mathrm{Aut}(\mathscr{D}, \sigma) = \{\Phi \in \mathrm{Aut}(\mathscr{D}) \mid \Phi(\sigma) = \sigma\}$.

Our study fits into the more general framework of classifying autoequivalences via their actions on the space of stability conditions. What we deal with in this paper is about autoequivalences of finite order. Beyond this, one can study autoequivalences from a dynamical perspective, for example, from the properties of being *reducible* or *pseudo-Anosov* (cf. [DHKK14, FFH+21]) when acting on certain compactifications of the space of stability conditions. Such results have been obtained in the case of elliptic curves by Kikuta–Koseki–Ouchi [KKO22]. In this paper, we classify autoequivalences of K3 surfaces of Picard number one into four types: *finite order*, (-2)*-reducible*, 0*-reducible*, and *pseudo-Anosov*, in terms of their actions on the punctured hyperbolic domain $\mathcal{Q}_0^+(X)$. We also propose some open questions regarding the reducible autoequivalences.

Organization of the paper In Section 2, we first recall basic notions about Bridgeland stability conditions on triangulated categories. Then we solve the realization problem for curves and standard autoequivalences on surfaces. In Section 3, as preparation for later sections, we first review fundamental properties about autoequivalences and stability conditions on K3 surfaces with arbitrary Picard numbers, then we prove the finiteness of Gepner type autoequivalences in this case. The realization problem for twisted abelian surfaces and generic twisted K3 surfaces will be answered in this section.

In Section 4, we compute the images of spherical twists in $\Gamma_0^+(n)$. Then we compute the numbers of elliptic points, cusps, and the holes corresponding to spherical objects on the quotient $\Gamma_0^+(n)\backslash\backslash\mathcal{Q}_0^+(X)$. This will be used to prove Theorem 1.2. The realization problem for K3 surfaces of Picard number one will also be solved in this section. In Section 5, we prove Theorems 1.1 and describe the distribution of Gepner type points on $\mathrm{Stab}^\dagger(X)$. We also discuss a classification of autoequivalences according to their actions on $\mathcal{Q}_0^+(X)$. Section 6 is devoted to characterizations for the existence of associated cubic fourfolds.

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2 Curves and standard autoequivalences on surfaces

In this section, we recall the definition of Bridgeland stability conditions on triangulated categories and certain natural group actions on the space of stability conditions. As a warm-up for later sections, we show that the categorical Nielsen realization problems have affirmative answers for bounded derived categories of coherent sheaves on curves, as well as the group of standard autoequivalences on surfaces. We also provide a classification of Gepner type autoequivalences in the curve case.

2.1 Preliminaries on stability conditions Let us review the definition of Bridgeland stability conditions following [Bri07]. Let $\mathscr{D}$ be a triangulated category, $K_0(\mathscr{D})$ be the Grothendieck group of $\mathscr{D}$, and $\mathrm{cl} : K_0(\mathscr{D}) \longrightarrow \Gamma$ be a group homomorphism to a finite rank abelian group Γ. Then a *Bridgeland stability condition* $\sigma = (Z, \mathscr{P})$ on $\mathscr{D}$ consists of

- a group homomorphism $Z \colon \Gamma \longrightarrow \mathbb{C}$ called the *central charge*, and
- a collection of full additive subcategories $\mathscr{P} = \{\mathscr{P}(\phi) \subseteq \mathscr{D}\}_{\phi \in \mathbb{R}}$ call the *slicing*

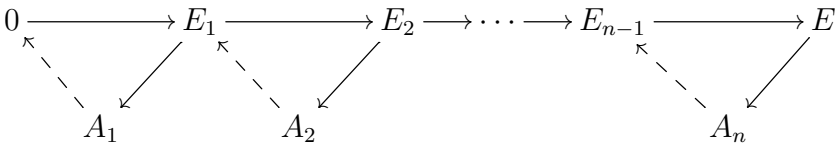
such that

(1) for every nonzero object $E \in \mathscr{P}(\phi)$, we have $Z(E) \coloneqq Z(\mathrm{cl}([E])) \in \mathbb{R}_{>0} \cdot e^{i\pi\phi}$;

(2) $\mathscr{P}(\phi + 1) = \mathscr{P}(\phi)[1]$,

(3) if $\phi_1 > \phi_2$ and $A_i \in \mathscr{P}(\phi_i)$ for $i = 1, 2$, then $\mathrm{Hom}(A_1, A_2) = 0$,

(4) for any nonzero object $E \in \mathscr{D}$, there exists a (unique) collection of exact triangles



where $A_i \in \mathscr{P}(\phi_i)$ and $\phi_1 > \phi_2 > \cdots > \phi_n$,

(5) there exists a constant $C > 0$ and a norm $||\cdot||$ on $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ such that

$$||\mathrm{cl}([E])|| \leq C|Z(E)| \qquad \text{for all} \qquad E \in \mathscr{P}(\phi).$$

Due to [Bri07, Proposition 5.3], giving a stability condition $\sigma = (Z, \mathscr{P})$ is equivalent to giving a pair $(Z, \mathscr{A})$, where $\mathscr{A}$ is the heart of a bounded t-structure on $\mathscr{D}$, such that the following additional conditions are satisfied:

(a) for any nonzero object $E \in \mathscr{A}$, we have $Z(E) \in \mathcal{H} \cup \mathbb{R}_{<0}$ where

$$\mathcal{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}.$$

(b) for any nonzero object $E \in \mathscr{A}$, there exists a (unique) filtration

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E \qquad \text{where} \qquad E_i \in \mathscr{A}$$

such that $A_i := E_i/E_{i-1}$ is Z-semistable for all i and $\arg(A_1) > \cdots > \arg(A_n)$. Here $A \in \mathscr{A}$ is called *Z-semistable* if $\arg(A) \geq \arg(B)$ for any subobject B of A in $\mathscr{A}$.

(c) there exists a constant $C > 0$ and a norm $||\cdot||$ on $\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ such that

$$||\mathrm{cl}([E])|| \leq C|Z(E)|$$

for any Z-semistable $E \in \mathscr{A}$.

We will be mainly considering the case where $\mathscr{D} = \mathrm{D^b}(X)$ is the bounded derived category of coherent sheaves on a smooth complex projective variety X. In this case Γ is chosen to be the numerical Grothendieck group

$$\mathcal{N}(\mathscr{D}) := K_0(\mathscr{D})/K_0(\mathscr{D})^{\perp\chi}$$

defined as the quotient of the Grothendieck group by the kernel of its Euler pairing

$$\chi(E_1, E_2) = \sum_i (-1)^i \dim \mathrm{Hom}(E_1, E_2[i]).$$

In this case, the homomorphism $\mathrm{cl}\colon K_0(\mathscr{D}) \longrightarrow \mathcal{N}(\mathscr{D})$ is the natural quotient.

Example 2.1. Let $\mathscr{D} = \mathrm{D^b}(X)$ be the bounded derived category of coherent sheaves on a smooth complex projective curve. The numerical Grothendieck group $\mathcal{N}(\mathscr{D})$ can be identified with $\mathbb{Z} \oplus \mathbb{Z}$ with the quotient map $K_0(\mathscr{D}) \longrightarrow \mathcal{N}(\mathscr{D})$ sending $[E]$ to $(\mathrm{rank}(E), \deg(E))$. Let us exhibit an example of a stability condition on $\mathscr{D}$: For any $\beta + i\omega \in \mathcal{H}$, define a group homomorphism $Z_{\beta,\omega}\colon \mathcal{N}(\mathscr{D}) \longrightarrow \mathbb{C}$ by

$$Z_{\beta,\omega}(E) = -\deg(E) + (\beta + i\omega)\,\mathrm{rank}(E).$$

For $\phi \in (0, 1]$, define $\mathscr{P}(\phi) \subseteq \mathscr{D}$ to be the subcategory with objects consisting of slope semistable coherent sheaves E with $Z_{\beta,\omega}(E) \in \mathbb{R}_{>0} \cdot e^{i\pi\phi}$. For the other $\phi \in \mathbb{R}$, define $\mathscr{P}(\phi)$ via the property $\mathscr{P}(\phi + 1) = \mathscr{P}(\phi)[1]$. Then $\sigma_{\beta,\omega} = (Z_{\beta,\omega}, \mathscr{P})$ gives a stability condition called a *geometric stability condition*.

Let $\mathrm{Stab}(\mathscr{D})$ be the set of Bridgeland stability conditions on $\mathscr{D}$. Bridgeland introduced a topology on $\mathrm{Stab}(\mathscr{D})$ and proves that the forgetful map

$$\mathrm{Stab}(\mathscr{D}) \longrightarrow \mathrm{Hom}(\Gamma, \mathbb{C}) : \sigma = (Z, \mathscr{P}) \longmapsto Z$$

is a local homeomorphism. This equips $\mathrm{Stab}(\mathscr{D})$ with the structure of a finite dimensional complex manifold. The space $\mathrm{Stab}(\mathscr{D})$ carries a left action by the group $\mathrm{Aut}(\mathscr{D})$ where an autoequivalence Φ acts as

$$\sigma = (Z, \mathscr{P}) \longmapsto \Phi(\sigma) := (Z \circ [\Phi]^{-1}, \mathscr{P}') \quad \text{where} \quad \mathscr{P}'(\phi) = \Phi(\mathscr{P}(\phi)).$$

There is also a right action by $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, namely, the universal cover of $\mathrm{GL}^+(2, \mathbb{R})$. An element of this group corresponds to a pair (T, f) where

- $T \in \mathrm{GL}^+(2,\mathbb{R})$, and
- $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing function that satisfies $f(\phi+1) = f(\phi)+1$,

such that the actions of f and T on the quotients $\mathbb{R}/2\mathbb{Z}$ and $(\mathbb{R}^2 \setminus \{0\})/\mathbb{R}_{>0}$, respectively, are compatible via the isomorphisms

$$\begin{array}{ccccc} \mathbb{R}/2\mathbb{Z} & \xrightarrow{\sim} & S^1 & \xrightarrow{\sim} & (\mathbb{R}^2 \setminus \{0\})/\mathbb{R}_{>0} \\ \phi & \longmapsto & e^{i\pi\phi}. & & \end{array}$$

Each (T, f) acts on $\mathrm{Stab}(\mathscr{D})$ by

$$\sigma = (Z, \mathscr{P}) \longmapsto \sigma \cdot g := (T^{-1} \circ Z, \mathscr{P}'') \quad \text{where} \quad \mathscr{P}''(\phi) := \mathscr{P}(f(\phi)).$$

The left action by $\mathrm{Aut}(\mathscr{D})$ and the right action by $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ commute with each other according to [Bri07, Lemma 8.2].

If we view $\mathbb{R}^2$ as the complex plane $\mathbb{C}$, then multiplications by complex numbers realizes $\mathbb{C}^*$ as a subgroup of $\mathrm{GL}^+(2,\mathbb{R})$, which lifts to a subgroup $\mathbb{C} \subseteq \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$. Each $\lambda \in \mathbb{C}$ acts on the space $\mathrm{Stab}(\mathscr{D})$ by

$$\sigma = (Z, \mathscr{P}) \longmapsto \sigma \cdot \lambda := (Z \cdot e^{-i\pi\lambda}, \mathscr{P}'') \quad \text{where} \quad \mathscr{P}''(\phi) = \mathscr{P}(\phi + \mathrm{Re}(\lambda)).$$

Notice that $\mathbb{C}$ is not a normal subgroup of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$. Indeed, we have the following elementary fact:

Lemma 2.2. *Pick $\lambda \in \mathbb{C}\setminus\mathbb{Z} \subseteq \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$. Then for every $g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ we have*

$$g^{-1}\lambda g \in \mathbb{C} \implies g \in \mathbb{C}.$$

Proof. Let $\overline{\lambda}$ and $\overline{g}$ be the images of λ and g under the projection

$$\widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \longrightarrow \mathrm{GL}^+(2,\mathbb{R}).$$

Without loss of generality, we can rescale them so that $\det(\overline{\lambda}) = \det(g) = 1$. Then

$$\overline{\lambda} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \text{and} \qquad \overline{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $\theta \notin \mathbb{Z}\pi$ and $ad - bc - 1$. One can check by a direct computation that $g^{-1}\lambda g \in \mathbb{C}$ if and only if

$$ab + cd = 0 \qquad \text{and} \qquad a^2 + c^2 = b^2 + d^2.$$

That is, the columns of $\overline{g}$, viewed as vectors in $\mathbb{R}^2$, are orthogonal of the same length under the Euclidean norm. This implies that $g \in \mathbb{C}$. □

The following lemma shows that an affirmative answer to the realization problem modulo even shifts gives an affirmative answer to the one without modulo even shifts.

Lemma 2.3. *Let $\mathscr{D}$ be a triangulated category with a stability condition σ and let G be a finite subgroup of $\mathrm{Aut}(\mathscr{D})$. If G preserves the set $\sigma \cdot \mathbb{C}$, then it fixes σ. Therefore, if a finite subgroup of $\mathrm{Aut}(\mathscr{D})$ fixes a point on $\mathrm{Stab}(\mathscr{D})/\mathbb{C}$, then it fixes a point on $\mathrm{Stab}(\mathscr{D})$.*

Proof. For every $\Phi \in G$, there exists $\lambda \in \mathbb{C}$ such that $\Phi(\sigma) = \sigma \cdot \lambda$, which implies that $\Phi^n(\sigma) = \sigma \cdot (n\lambda)$ for all integer n. Since Φ is of finite order, one obtains $n\lambda = 0$ for some positive n. Thus $\lambda = 0$ and $\Phi(\sigma) = \sigma$. □

2.2 Curves and their Gepner type autoequivalences Let X be a smooth complex projective curve. Then its group of autoequivalences acts on the numerical Grothendieck group $\mathcal{N}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ as $\mathrm{SL}(2,\mathbb{Z})$, where the shift functor [1] acts as $-\mathrm{id}$. Consider the composition

$$\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow \mathrm{SL}(2,\mathbb{Z}) \longrightarrow \mathrm{PSL}(2,\mathbb{Z})$$

and denote its kernel as $\widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$. Then

$$\widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X)) \cong (\mathrm{Pic}^0(X) \rtimes \mathrm{Aut}(X)) \times \mathbb{Z}[1].$$

Note that a stability condition $\sigma_{\beta,\omega} \in \mathrm{Stab}(X)$ given as in Example 2.1 is preserved under the action of $\mathrm{Pic}^0(X) \rtimes \mathrm{Aut}(X)$. This means that every autoequivalence in $\widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$ is of Gepner type.

Proposition 2.4. *Let X be a smooth complex projective curve. Then*

- *every finite subgroup of* $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ *fixes a point on* $\mathrm{Stab}(X)/\mathbb{C}$*, and*
- *every finite subgroup of* $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ *fixes a point on* $\mathrm{Stab}(X)$*.*

Proof. Let us first deal with the case when X has genus $g \neq 1$. In this case,

$$\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \cong (\mathrm{Pic}(X) \rtimes \mathrm{Aut}(X)) \times \mathbb{Z}[1].$$

This induces a short exact sequence

$$0 \longrightarrow \widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2] \longrightarrow \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2] \longrightarrow \mathbb{Z} \longrightarrow 0.$$

As there is no finite subgroup in $\mathbb{Z}$, every finite subgroup of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ is contained in the kernel $\widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$, thus preserves $\overline{\sigma}_{\beta,\omega} \in \mathrm{Stab}(X)/\mathbb{C}$ for every $\sigma_{\beta,\omega}$ constructed as in Example 2.1.

Now assume that X has genus one. By [Bri07, Theorem 9.1], there is an isomorphism $\mathrm{Stab}(X) \cong \mathbb{C} \times \mathcal{H}$. In addition, the quotient $\mathrm{Stab}(X)/\mathbb{C} \cong \mathcal{H}$ admits a section in $\mathrm{Stab}(X)$ given by stability conditions $\sigma_{\beta,\omega}$ constructed in Example 2.1. It is straightforward to check that the action of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ on $\mathrm{Stab}(X)/\mathbb{C} \cong \mathcal{H}$ is compatible with the standard $\mathrm{SL}(2,\mathbb{Z})$-action on $\mathcal{H}$ through the homomorphism $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2] \longrightarrow \mathrm{SL}(2,\mathbb{Z})$. The statement then follows from the fact that every finite subgroup of $\mathrm{SL}(2,\mathbb{Z})$ fixes a point, called an *elliptic point*, on $\mathcal{H}$.

This proves the first statement. The second statement follows immediately from the first one due to Lemma 2.3. □

Remark 2.5. The converse of Proposition 2.4 is not true, that is, a Gepner type autoequivalence on a curve X may not be of finite order in $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$. For instance, an autoequivalence induced by an infinite order automorphism of X is of Gepner type, and is of infinite order in $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$. In contrast, we will show in later sections that for K3 surfaces of Picard number one, an autoequivalence is of Gepner type if and only if it is of finite order in $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$.

To give a classification of Gepner type autoequivalences, let us first characterize autoequivalences which are of Gepner type with respect to geometric stability conditions. A stability condition $\sigma \in \mathrm{Stab}(X)$ is called *geometric* if all skyscraper sheaves are σ-stable of the same phase. Let $U(X) \subseteq \mathrm{Stab}(X)$ be the set of geometric stability conditions. By [Mac07, Theorem 2.7] and [BMW15, Section 3.2], there is an isomorphism

$$\mathcal{H} \times \mathbb{C} \xrightarrow{\sim} U(X) : (\beta + i\omega, \lambda) \longmapsto \sigma_{\beta,\omega} \cdot \lambda.$$

Recall that the $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$-action on $U(X)/\mathbb{C} \cong \mathcal{H}$ is compatible with the standard $\mathrm{SL}(2,\mathbb{Z})$-action on $\mathcal{H}$. Therefore, $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ is of Gepner type with respect to a geometric stability condition if and only if the action of $[\Phi] \in \mathrm{SL}(2,\mathbb{Z})$ on $\mathcal{H}$ has a fixed point, which is equivalent to $[\Phi]$ being of finite order.

Before proceeding further, let us introduce some necessary notations by looking at a few examples about how the map $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow \mathrm{SL}(2,\mathbb{Z})$ works. First, for curves of any genus, the map sends $T := - \otimes \mathcal{O}_X(1)$ to

$$[T] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

For curves of genus one, let $P \in \mathrm{Coh}(X \times X)$ be the Poincaré line bundle. Then the Fourier–Mukai transform $S := \Phi_P$ along P is mapped to

$$[S] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Recall that every non-identity finite order element of $\mathrm{PSL}(2,\mathbb{Z})$, called an *elliptic element*, is conjugate to either $[S]$, $[ST]$, or $[(ST)^2]$.

Lemma 2.6. *Let X be a smooth complex projective curve of genus g and Φ be an autoequivalence on $\mathrm{D}^{\mathrm{b}}(X)$. Then the condition that Φ is of Gepner type with respect to a geometric stability condition is equivalent to one of the followings depending on g:*

- *If $g \neq 1$, then the condition is equivalent to $\Phi \in \widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$.*
- *If $g = 1$, then the condition is equivalent to the existence of $\Psi \in \widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$ such that $\Phi\Psi$ is conjugate to either $S, ST, (ST)^2$, or the identity.*

Proof. Recall that Φ is of Gepner type with respect to a geometric stability condition if and only if the action of $[\Phi] \in \mathrm{SL}(2,\mathbb{Z})$ on $\mathcal{H}$ is of finite order.

When $g \neq 1$, the image of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow \mathrm{SL}(2,\mathbb{Z})$ is generated by $[T]$ and $[[1]] = -\mathrm{id}$. In this case, $[\Phi]$ is of finite order if and only if $[\Phi] = \pm\mathrm{id}$, where the latter condition is equivalent to $\Phi \in \widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$. This proves the statement when $g \neq 1$.

When $g = 1$, the map $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow \mathrm{SL}(2,\mathbb{Z})$ is surjective, so $[\Phi]$ is of finite order if and only if it is conjugate to either $\pm[S]$, $\pm[ST]$, $\pm[(ST)^2]$, or $\pm\mathrm{id}$, which is equivalent to the existence of $\Psi \in \widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$ such that $\Phi\Psi$ is conjugate to either $S, ST, (ST)^2$, or the identity. □

Proposition 2.7. *Let X be a smooth complex projective curve of genus g and Φ be an autoequivalence on $\mathrm{D}^{\mathrm{b}}(X)$.*

- *If $g \neq 1$, then Φ is of Gepner type if and only if $\Phi \in \widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$.*
- *If $g = 1$, then Φ is of Gepner type if and only if there exists $\Psi \in \widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$ such that $\Phi\Psi$ is conjugate to either $S, ST, (ST)^2$, or the identity.*

Proof. By [Bri07, Theorem 0.1] and [Mac07, Theorem 2.7], any stability condition on $\mathrm{D}^{\mathrm{b}}(X)$ is geometric if $g \geq 1$. This completes the proof in this case by Lemma 2.6.

Now we consider the case $X = \mathbb{P}^1$ where non-geometric stability conditions exist. As every autoequivalence in $\widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1))$ is of Gepner type by Lemma 2.6, it remains to prove that every $\Phi \notin \widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1))$ is not of Gepner type, and it suffices to prove that it is not of Gepner type with respect to non-geometric stability conditions. Notice that every such Φ is of the form

$$\Phi = f^*(- \otimes \mathcal{O}(k))[n] \qquad (k \neq 0)$$

for some $f \in \mathrm{Aut}(\mathbb{P}^1)$ and $n \in \mathbb{Z}$. By [Oka06] and [BMW15, Section 3.2], if $\sigma \in \mathrm{Stab}(\mathbb{P}^1)$ is non-geometric, then there exists $\ell \in \mathbb{Z}$ such that $\mathcal{O}(\ell)$, $\mathcal{O}(\ell+1)$, and their shifts are the only σ-stable objects. Because $k \neq 0$, the autoequivalence Φ does not preserve the set of σ-stable objects for non-geometric σ, while the free $\mathbb{C}$-action on σ does preserve the set of σ-stable objects. Thus Φ is not of Gepner type with respect to non-geometric stability conditions. This concludes the proof. □

2.3 Standard autoequivalences on surfaces Let us review the tilting constructions of stability conditions on smooth projective complex surfaces. First, recall that a *torsion pair* on an abelian category $\mathscr{A}$ consists of a pair of full additive subcategories $(\mathscr{T}, \mathscr{F})$ such that:

(a) for any $T \in \mathscr{T}$ and $F \in \mathscr{F}$, we have $\mathrm{Hom}(T, F) = 0$;

(b) for any $E \in \mathscr{A}$, there exist (unique) $T \in \mathscr{T}$ and $F \in \mathscr{F}$ together with a short exact sequence $0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0$.

Lemma 2.8 ([HRS96])**.** *Let $\mathscr{A}$ be the heart of a bounded t-structure on a triangulated category $\mathscr{D}$ and $(\mathscr{T}, \mathscr{F})$ be a torsion pair on $\mathscr{A}$. Then the category*

$$\mathscr{A}^\sharp = \left\{ E \in \mathscr{D} \mid H^0_{\mathscr{A}}(E) \in \mathscr{T},\ H^{-1}_{\mathscr{A}}(E) \in \mathscr{F},\ H^i_{\mathscr{A}}(E) = 0 \ \text{ for } \ i \neq 0, -1 \right\}$$

is the heart of a bounded t-structure on $\mathscr{D}$. Here $H^\bullet_{\mathscr{A}}$ denotes the cohomology object with respect to the t-structure of $\mathscr{A}$.

Let X be a smooth complex projective surface, in which case the numerical Grothendieck group of $\mathrm{D}^{\mathrm{b}}(X)$ is given by

$$\mathcal{N}(X) \cong H^0(X, \mathbb{Z}) \oplus \mathrm{NS}(X) \oplus H^4(X, \mathbb{Z}).$$

Let $\beta, \omega \in \mathrm{NS}(X) \otimes \mathbb{R}$ be a pair of $\mathbb{R}$-divisors such that ω is ample. Then the slope function

$$\mu_\omega(E) = \frac{\omega \cdot c_1(E)}{\mathrm{rank}(E)}$$

defines a (slope) stability on the abelian category $\mathscr{A} = \mathrm{Coh}(X)$. Each $E \in \mathrm{Coh}(X)$ has a unique Harder–Narasimhan filtration

$$0 \subseteq E_{\mathrm{tor}} = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_n = E$$

where E_{tor} is the torsion part of E, and each E_i/E_{i-1} is a torsion-free μ_ω-semistable sheaf of slope μ_i with $\mu_\omega^{\max}(E) = \mu_1 > \cdots > \mu_n = \mu_\omega^{\min}(E)$. Then

$$\mathscr{T}_{\beta,\omega} = \{\text{torsion sheaves}\} \cup \{E \mid \mu_\omega^{\min}(E) > \beta \cdot \omega\} \quad \text{and} \quad \mathscr{F}_{\beta,\omega} = \{E \mid \mu_\omega^{\max}(E) \leq \beta \cdot \omega\}$$

form a torsion pair on $\mathscr{A}$. One obtains a new heart $\mathscr{A}^\sharp_{\beta,\omega} \subseteq \mathrm{D^b}(X)$ by applying Lemma 2.8 to this torsion pair.

Theorem 2.9 ([AB13, Corollary 2.1])**.** *Let X be a smooth complex projective surface. Consider the central charge*

$$Z'_{\beta,\omega}(E) = e^{\beta+i\omega} \cdot \mathrm{ch}(E).$$

Then the pair $\sigma'_{\beta,\omega} = (Z'_{\beta,\omega}, \mathscr{A}^\sharp_{\beta,\omega})$ gives a stability condition on $\mathrm{D^b}(X)$.

Let X be a smooth complex projective variety. Then the elements in the subgroup

$$\mathrm{Aut}_{\mathrm{std}}(\mathrm{D^b}(X)) := (\mathrm{Pic}(X) \rtimes \mathrm{Aut}(X)) \times \mathbb{Z}[1] \subseteq \mathrm{Aut}(\mathrm{D^b}(X))$$

are called *standard autoequivalences*. The following proposition shows that the realization problem has a positive answer for standard autoequivalences on a surface.

Proposition 2.10. *Let X be a smooth complex projective surface. Then*

- *every finite subgroup of $\mathrm{Aut}_{\mathrm{std}}(\mathrm{D^b}(X))/\mathbb{Z}[2]$ fixes a point on $\mathrm{Stab}(X)/\mathbb{C}$, and*
- *every finite subgroup of $\mathrm{Aut}_{\mathrm{std}}(\mathrm{D^b}(X))$ fixes a point on $\mathrm{Stab}(X)$.*

Proof. Notice that $[1]$ fixes $\mathrm{Stab}(X)/\mathbb{C}$ pointwisely. To prove both statements, it suffices to prove that every finite subgroup of $\mathrm{Pic}(X) \rtimes \mathrm{Aut}(X)$ fixes a point on $\mathrm{Stab}(X)$. In fact, we will show that such a subgroup fixes the stability condition $\sigma'_{\beta,\omega} = (Z'_{\beta,\omega}, \mathscr{A}^\sharp_{\beta,\omega})$ given in Theorem 2.9.

First we show that, for every $L \in \mathrm{Pic}(X)$ such that $L^{\otimes n} \cong \mathcal{O}_X$ for some $n \geq 1$, the autoequivalence $- \otimes L$ fixes $\sigma'_{\beta,\omega} = (Z'_{\beta,\omega}, \mathscr{A}^\sharp_{\beta,\omega})$. It is sufficient to show that it preserves the slope function μ_ω and the central charge $Z'_{\beta,\omega}$. Notice that $nc_1(L) = 0$. Thus,

$$n\left(\mu_\omega(E \otimes L) - \mu_\omega(E)\right) = \frac{n(c_1(E \otimes L) - c_1(E)) \cdot \omega}{\mathrm{rank}(E)} = nc_1(L) \cdot \omega = 0$$

which implies $\mu_\omega(E \otimes L) = \mu_\omega(E)$. Similarly,

$$\begin{aligned} n\left(Z'_{\beta,\omega}(E \otimes L) - Z'_{\beta,\omega}(E)\right) &= ne^{\beta+i\omega} \cdot (\mathrm{ch}(E \otimes L) - \mathrm{ch}(E)) \\ &= -nc_1(L) \cdot \mathrm{ch}_1(E) - \frac{1}{2}nc_1(L)^2 \cdot \mathrm{ch}_0(E) = 0. \end{aligned}$$

Hence $Z'_{\beta,\omega}(E \otimes L) = Z'_{\beta,\omega}(E)$. As a result, every finite subgroup of $\mathrm{Pic}(X)$ fixes $\sigma'_{\beta,\omega}$.

It remains to show that any finite subgroup $G \subseteq \mathrm{Aut}(X)$ fixes $\sigma'_{\beta,\omega}$. Choose any ample divisor $\alpha \in \mathrm{Amp}(X)$ and take the average

$$\widetilde{\alpha} = \sum_{g \in G} g^* \alpha.$$

Then $\widetilde{\alpha}$ is a G-invariant ample divisor. Let β (resp. ω) to be a real (resp. positive) multiple of $\widetilde{\alpha}$. It is clear that the slope function μ_ω, the torsion pair $(\mathscr{T}_{\beta,\omega}, \mathscr{F}_{\beta,\omega})$, and the central charge $Z'_{\beta,\omega}$ are all G-invariant. This implies that $\sigma'_{\beta,\omega}$ is fixed by G. □

Remark 2.11. Given a complex algebraic K3 surface X, consider the Mukai vector

$$v = \mathrm{ch} \cdot \sqrt{\mathrm{td}_X} = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_0 + \mathrm{ch}_2)$$

and the central charge

$$Z_{\beta,\omega}(E) = e^{\beta + i\omega} \cdot v(E).$$

Then the pair $\sigma_{\beta,\omega} = (Z_{\beta,\omega}, \mathscr{A}^\sharp_{\beta,\omega})$ gives a stability condition on $\mathrm{D^b}(X)$, provided β and ω are chosen so that for all spherical sheaf E on X one has $Z_{\beta,\omega}(E) \notin \mathbb{R}_{\leq 0}$. This holds in particular when $\omega^2 > 2$ [Bri08, Lemma 6.2]. (This can be attained by choosing ω to be a sufficiently large multiple of an invariant ample class.) Following the same strategy of proof, one can reproduce Proposition 2.10 for K3 surfaces using such a stability condition.

Remark 2.12. One can prove that Nielsen realization holds for standard autoequivalences in the same manner for $\dim(X) \geq 3$, so long as there is a construction of Bridgeland stability conditions on $\mathrm{D^b}(X)$ which involves only the choice of an ample class ω. For instance, this applies to smooth Fano threefolds and quintic threefolds, by the works of [BMSZ17, Li19].

3 Gepner type autoequivalences on K3 surfaces

Gepner type autoequivalences on K3 surfaces can be described in a more explicit way in terms of their actions on certain period domains of stability conditions. This section is mainly devoted to the necessary preliminaries and a general discussion on Gepner type autoequivalences in the case of K3 surfaces. Based on the terminology established in the preliminaries, we discuss in the last part of this section the realization problem for abelian surfaces and generic twisted K3 surfaces.

3.1 Basics on autoequivalences and stability conditions

Let us review basic facts about autoequivalences and stability conditions on K3 surfaces after [Bri08, HMS09]. Along the way, we will set up notations to be used in the remaining part of this paper.

Given an algebraic K3 surface X, one can extend the polarized weight two Hodge structure on the middle cohomology $H^2(X, \mathbb{Z})$ to the total cohomology

$$\widetilde{H}(X, \mathbb{Z}) := H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$$

such that the $(1,1)$-part is isomorphic to the numerical Grothendieck group

$$\mathcal{N}(X) \cong H^0(X, \mathbb{Z}) \oplus \mathrm{NS}(X) \oplus H^4(X, \mathbb{Z})$$

and the polarization is obtained by extending the intersection product on $H^2(X,\mathbb{Z})$ orthogonally to the other summands as

$$(r_1, D_1, s_1) \cdot (r_2, D_2, s_2) = D_1 \cdot D_2 - r_1 s_2 - s_1 r_2.$$

This turns $\widetilde{H}(X,\mathbb{Z})$ into an even unimodular lattice of signature $(4,20)$ called the *Mukai lattice*. The orientations of two given positive 4-planes in $\widetilde{H}(X,\mathbb{R}) := \widetilde{H}(X,\mathbb{Z}) \otimes \mathbb{R}$ are either coincide or opposite to each other after projected orthogonally from an arbitrary negative 20-plane. In particular, one can compare the orientations of a positive 4-plane and its image under a Hodge isometry $\phi \in \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}))$ this way. Let us denote by

$$\mathrm{Aut}^+(\widetilde{H}(X,\mathbb{Z})) \subseteq \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}))$$

the subgroup of orientation-preserving isometries.

As asserted by [HMS09, Corollary 3], every autoequivalence $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ induces an element $\phi \in \mathrm{Aut}^+(\widetilde{H}(X,\mathbb{Z}))$ and, conversely, every element of $\mathrm{Aut}^+(\widetilde{H}(X,\mathbb{Z}))$ arises this way. This gives a short exact sequence

$$0 \longrightarrow \mathcal{I}(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow \mathrm{Aut}^+(\widetilde{H}(X,\mathbb{Z})) \longrightarrow 0. \tag{3.1}$$

and we call the kernel $\mathcal{I}(\mathrm{D}^{\mathrm{b}}(X))$ the *Torelli group*. The notation and name for the kernel are inspired by the study of mapping class groups $\mathrm{Mod}(\Sigma)$ of topological surfaces Σ, where the subgroup of $\mathrm{Mod}(\Sigma)$ acting trivially on $H_1(\Sigma,\mathbb{Z})$ is called the Torelli group and usually denoted as $\mathcal{I}(\Sigma)$. Let us denote by

$$\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X)) \subseteq \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \qquad \text{and} \qquad \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z})) \subseteq \mathrm{Aut}^+(\widetilde{H}(X,\mathbb{Z}))$$

the subgroups of symplectic elements, that is, the elements acting trivially on $H^{2,0}(X)$. Then the exact sequence (3.1) contains the exact subsequence

$$0 \longrightarrow \mathcal{I}(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow \mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow \mathrm{Aut}_s^+(\widetilde{H}(X,\mathbb{Z})) \longrightarrow 0.$$

Notice that $\mathcal{I}(\mathrm{D}^{\mathrm{b}}(X))$ appears as the kernel as its elements are symplectic by definition.

Every stability condition $\sigma = (Z, \mathscr{P})$ on X uniquely determines a $w \in \mathcal{N}(X) \otimes \mathbb{C}$ such that the central charge takes the form $Z(-) = (w, v(-))$, where $v(-) = \mathrm{ch}(-)\sqrt{\mathrm{td}(X)}$ is the Mukai vector. This identifies Z as an element of $\mathcal{N}(X) \otimes \mathbb{C}$ and thus induces a map

$$\mathrm{Stab}(X) \longrightarrow \mathcal{N}(X) \otimes \mathbb{C} : (Z, \mathscr{P}) \longmapsto Z. \tag{3.2}$$

There is a distinguished connected component $\mathrm{Stab}^\dagger(X) \subseteq \mathrm{Stab}(X)$ which contains the set of stability conditions for which all skyscraper sheaves $\mathcal{O}_x$ of points $x \in X$ are stable of the same phase. To describe the image of (3.2) when restricted to this component, let us first consider the open subset

$$\mathcal{P}(X) := \{\gamma \in \mathcal{N}(X) \otimes \mathbb{C} \mid \mathrm{Re}(\gamma), \mathrm{Im}(\gamma) \text{ span a positive plane in } \mathcal{N}(X) \otimes \mathbb{R}\}.$$

The real and imaginary parts of each $\gamma \in \mathcal{P}(X)$ determine a natural orientation on the positive plane spanned by them. This leads to a decomposition into connected components

$$\mathcal{P}(X) = \mathcal{P}^+(X) \sqcup \mathcal{P}^-(X)$$

which are complex conjugate to each other. Here we require $\mathcal{P}^+(X)$ to be the component which contains e^{ih} with $h \in \mathrm{NS}(X)$ an ample class.

By Bridgeland [Bri08, Theorem 1.1], if we remove every hyperplane section on $\mathcal{P}^+(X)$ which is orthogonal to a (-2)-vector in $\mathcal{N}(X)$ to get

$$\mathcal{P}^+_0(X) := \mathcal{P}^+(X) \setminus \bigcup_{\delta \in \mathcal{N}(X),\, \delta^2 = -2} \delta^\perp,$$

then the restriction of (3.2) to the component $\mathrm{Stab}^\dagger(X)$ defines a topological covering

$$\pi \colon \mathrm{Stab}^\dagger(X) \longrightarrow \mathcal{P}^+_0(X).$$

Moreover, the subgroup $\mathcal{I}^\dagger(\mathrm{D^b}(X)) \subseteq \mathcal{I}(\mathrm{D^b}(X))$ preserving the component $\mathrm{Stab}^\dagger(X)$ acts freely on $\mathrm{Stab}^\dagger(X)$ and is the group of deck transformations of π. Due to this result, the space $\mathcal{P}^+_0(X)$ may be considered as a *period domain* for stability conditions.

There is another space $\mathcal{Q}^+_0(X)$ which may also be referred to as a period domain. To define it, let us start with

$$\mathcal{Q}(X) := \{[\gamma] \in \mathbb{P}(\mathcal{N}(X) \otimes \mathbb{C}) \mid \gamma^2 = 0,\ \gamma\overline{\gamma} > 0\}. \tag{3.3}$$

For each class $[\gamma] \in \mathcal{Q}(X)$, the oriented plane in $\mathcal{N}(X) \otimes \mathbb{R}$ spanned by $\mathrm{Re}(\gamma)$ and $\mathrm{Im}(\gamma)$ is independent of the choices of representatives γ. This assignment identifies $\mathcal{Q}(X)$ as the open subset of positive planes within the Grassmannian of oriented planes. In particular, $\mathcal{Q}(X)$ has two connected components. Let us denote them as $\mathcal{Q}^+(X)$ and $\mathcal{Q}^-(X)$ so that they are respectively the bases under $\mathcal{P}^+(X)$ and $\mathcal{P}^-(X)$ in the $\mathrm{GL}^+(2,\mathbb{R})$-fibration

$$\mathcal{P}(X) \longrightarrow \mathcal{Q}(X) : \gamma \longmapsto \text{the oriented plane spanned by } \mathrm{Re}(\gamma) \text{ and } \mathrm{Im}(\gamma).$$

The space we desire is then obtained by removing the locus orthogonal to (-2)-vectors

$$\mathcal{Q}^+_0(X) := \mathcal{Q}^+(X) \setminus \bigcup_{\delta \in \mathcal{N}(X),\, \delta^2 = -2} \delta^\perp.$$

In the study of Gepner type autoequivalences for K3 surfaces, we will consider only the stability conditions lying on $\mathrm{Stab}^\dagger(X)$. The main reason is that, for K3 surfaces, this component is expected to be a simply connected space invariant under the actions of autoequivalences [Bri08, Conjecture 1.2], which is known to hold in the case of Picard number one [BB17, Theorem 1.3]. In view of this conjecture, the structure of the group of autoequivalences should be totally determined by how it acts on $\mathrm{Stab}^\dagger(X)$. Notice that, if an autoequivalence is of Gepner type with respect to a stability condition on $\mathrm{Stab}^\dagger(X)$, then it preserves $\mathrm{Stab}^\dagger(X)$ as the $\mathbb{C}$-action preserves connected components. Suppose that Φ is an

autoequivalence of Gepner type with respect to $\sigma = (Z, \mathscr{P}) \in \mathrm{Stab}^\dagger(X)$ and let ϕ denote its action on the Mukai lattice. Then there exists $\lambda \in \mathbb{C}$ such that

$$\Phi(\sigma) = \sigma \cdot \lambda \qquad \text{which implies} \qquad \phi(Z) = Z \cdot e^{-i\pi\lambda}.$$

Let $P \subseteq \mathcal{N}(X) \otimes \mathbb{R}$ be the positive oriented plane spanned by Z. Then the second relation shows that ϕ fixes P as a point on $\mathcal{Q}_0^+(X)$, and its action on $\mathcal{P}_0^+(X)$ is a change of frames on the plane P by applying $e^{-i\pi\lambda} \in \mathrm{GL}^+(2, \mathbb{R})$. This observation lies in the core of our study of Gepner type autoequivalences in the K3 case.

3.2 Groups of Gepner type autoequivalences Given an algebraic K3 surface X and a stability condition $\sigma \in \mathrm{Stab}^\dagger(X)$ with central charge Z, one can consider the group of autoequivalences on $\mathrm{D^b}(X)$ which fix σ:

$$\mathrm{Aut}(\mathrm{D^b}(X), \sigma) := \{\Phi \in \mathrm{Aut}(\mathrm{D^b}(X)) \mid \Phi(\sigma) = \sigma\}$$

and also the group of Hodge isometries on $\widetilde{H}(X, \mathbb{Z})$ which fix Z:

$$\mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}), Z) := \{\phi \in \mathrm{Aut}(\widetilde{H}(X, \mathbb{Z})) \mid \phi(Z) = Z\}.$$

Let us denote by $\mathrm{Aut}_s(\mathrm{D^b}(X), \sigma)$ and $\mathrm{Aut}_s(\widetilde{H}(X, \mathbb{Z}), Z)$ respectively the subgroups of symplectic elements in these two groups. By [Huy16b, Remark 1.2 & Proposition 1.4], these groups are finite and there is an isomorphism

$$\mathrm{Aut}(\mathrm{D^b}(X), \sigma) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}), Z). \tag{3.4}$$

This isomorphism also induces an isomorphism between their symplectic subgroups.

We are going to extend the above picture to Gepner type autoequivalences following the same strategy. Autoequivalences of Gepner type with respect to σ form a subgroup

$$\mathrm{Aut}(\mathrm{D^b}(X),\, \sigma \cdot \mathbb{C}) := \{\Phi \in \mathrm{Aut}(\mathrm{D^b}(X)) \mid \Phi(\sigma) = \sigma \cdot \lambda \ \text{ for some } \ \lambda \in \mathbb{C}\}.$$

Similarly, Hodge isometries which acts on Z as a rescaling form a subgroup

$$\mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}),\, Z \cdot \mathbb{C}^*) := \{\phi \in \mathrm{Aut}(\widetilde{H}(X, \mathbb{Z})) \mid \phi(Z) = Z \cdot \kappa \ \text{ for some } \ \kappa \in \mathbb{C}^*\}.$$

Let us denote by $\mathrm{Aut}_s(\mathrm{D^b}(X),\, \sigma \cdot \mathbb{C})$ and $\mathrm{Aut}_s(\widetilde{H}(X, \mathbb{Z}),\, Z \cdot \mathbb{C}^*)$ respectively the subgroups of symplectic elements in these two groups.

Lemma 3.1. *We have* $\mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}),\, Z \cdot \mathbb{C}^*) \subseteq \mathrm{Aut}^+(\widetilde{H}(X, \mathbb{Z}))$.

Proof. Pick any $\phi \in \mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}),\, Z \cdot \mathbb{C}^*)$ and let $P \subseteq \mathcal{N}(X) \otimes \mathbb{R}$ be the oriented positive plane determined by Z. The fact that $\phi(Z) = Z \cdot \kappa$ for some $\kappa \in \mathbb{C}^*$ implies that it preserves the orientation of P. Because ϕ preserves the Hodge structure, it also preserves the orientation of the positive plane $Q := (H^{2,0} \oplus H^{0,2})(X, \mathbb{R}) \subseteq \widetilde{H}(X, \mathbb{R})$. The two planes P and Q span a positive 4-plane in $\widetilde{H}(X, \mathbb{R})$ whose orientation in invariant under the action of ϕ, whence $\phi \in \mathrm{Aut}^+(\widetilde{H}(X, \mathbb{Z}))$. □

Lemma 3.2. *The symplectic subgroup* $\mathrm{Aut}_s(\widetilde{H}(X,\mathbb{Z}),\, Z\cdot\mathbb{C}^*)$ *is finite.*

Proof. Every element in this group is uniquely determined by its action on $\mathcal{N}(X)$, so we can identify it as a subgroup of

$$\mathrm{O}(\mathcal{N}(X),\, Z\cdot\mathbb{C}^*) := \{\phi\in \mathrm{O}(\mathcal{N}(X)) \mid \phi(Z) = Z\cdot\kappa \ \text{ for some } \ \kappa\in\mathbb{C}^*\}.$$

Let $P\subseteq \mathcal{N}(X)\otimes\mathbb{R}$ be the oriented positive plane determined by Z. Then restricting an isometry to P and its orthogonal complement $P^\perp\subseteq\mathcal{N}(X)\otimes\mathbb{R}$ defines an embedding

$$\mathrm{O}(\mathcal{N}(X),\, Z\cdot\mathbb{C}^*)\lhook\joinrel\longrightarrow \mathrm{O}(P)\times\mathrm{O}(P^\perp)$$

Because P and $P^\perp$ are respectively positive and negative definite, $\mathrm{O}(P)$ and $\mathrm{O}(P^\perp)$ are compact, whence $\mathrm{O}(P)\times\mathrm{O}(P^\perp)$ is compact by Tychonoff's theorem. The above embedding realizes $\mathrm{O}(\mathcal{N}(X),\, Z\cdot\mathbb{C}^*)$ as a discrete subgroup of a compact group, so it is finite. This implies that $\mathrm{Aut}_s(\widetilde{H}(X,\mathbb{Z}),\, Z\cdot\mathbb{C}^*)$ is finite. □

Lemma 3.3. *The group* $\mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),\, Z\cdot\mathbb{C}^*)$ *is finite.*

Proof. Let $T(X)$ be the transcendental lattice of X. Then $T(X)\otimes\mathbb{R}$ contains a positive plane $Q := (H^{2,0}\oplus H^{0,2})(X,\mathbb{R})$ with a negative orthogonal complement $Q^\perp\subseteq T(X)\otimes\mathbb{R}$. Restricting a Hodge isometry on $T(X)$ to Q and $Q^\perp$ defines an embedding

$$\mathrm{Aut}(T(X))\lhook\joinrel\longrightarrow \mathrm{O}(Q)\times\mathrm{O}(Q^\perp).$$

Notice that $\mathrm{O}(Q)$ and $\mathrm{O}(Q^\perp)$ are compact, whence $\mathrm{O}(Q)\times\mathrm{O}(Q^\perp)$ is compact as well. This realizes $\mathrm{Aut}(T(X))$ as a discrete subgroup of a compact group, thus it is finite.

Now, restricting a Hodge isometry on $\widetilde{H}(X,\mathbb{Z})$ to Q induces an exact sequence

$$0\longrightarrow \mathrm{Aut}_s(\widetilde{H}(X,\mathbb{Z}),\, Z\cdot\mathbb{C}^*)\longrightarrow \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),\, Z\cdot\mathbb{C}^*)\xrightarrow{|_Q}\mathrm{O}(Q).$$

The restriction map on the right has a finite image since it decomposes as

$$\mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),\, Z\cdot\mathbb{C}^*)\xrightarrow{|_{T(X)}}\mathrm{Aut}(T(X))\lhook\joinrel\longrightarrow \mathrm{O}(Q)\times\mathrm{O}(Q^\perp)\twoheadrightarrow \mathrm{O}(Q)$$

where the last surjection is the projection to the first factor. Lemma 3.2 shows that the kernel $\mathrm{Aut}_s(\widetilde{H}(X,\mathbb{Z}),\, Z\cdot\mathbb{C}^*)$ is finite, so $\mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),\, Z\cdot\mathbb{C}^*)$ is finite due to the exactness of the sequence. □

Proposition 3.4. *Let X be an algebraic K3 surface and $\sigma\in\mathrm{Stab}^\dagger(X)$ be a stability condition with central charge Z. Then the homomorphism*

$$\eta\colon\ \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X),\, \sigma\cdot\mathbb{C})\longrightarrow \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),\, Z\cdot\mathbb{C}^*)$$

which maps an autoequivalence to its action on the Mukai lattice is surjective with kernel equal to $\mathbb{Z}[2]$. In particular, η induces an isomorphism

$$\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X),\, \sigma\cdot\mathbb{C})\,/\,\mathbb{Z}[2]\xrightarrow{\sim}\mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),\, Z\cdot\mathbb{C}^*)$$

between finite groups. The same statement holds with Aut *replaced by* Aut_s.

Proof. Let us first show that η is surjective. Let $\phi \in \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),\, Z\cdot\mathbb{C}^*)$ and $\lambda\in\mathbb{C}$ be any element which satisfies $\phi(Z) = Z\cdot e^{-i\pi\lambda}$. It is known that the subgroup

$$\mathrm{Aut}^\dagger(\mathrm{D}^\mathrm{b}(X)) := \{\Phi\in\mathrm{Aut}(\mathrm{D}^\mathrm{b}(X)) \mid \Phi(\mathrm{Stab}^\dagger(X)) = \mathrm{Stab}^\dagger(X)\}$$

is surjective onto $\mathrm{Aut}^+(\widetilde{H}(X,\mathbb{Z}))$ [Huy16b, Section 1.3]. This fact, together with Lemma 3.1, implies that ϕ is induced by some $\Phi\in\mathrm{Aut}^\dagger(\mathrm{D}^\mathrm{b}(X))$. Both $\Phi(\sigma)$ and $\sigma\cdot\lambda$ belong to $\mathrm{Stab}^\dagger(X)$ and share the same central charge $Z\cdot e^{-i\pi\lambda}$, so there exists $\Psi\in\mathcal{I}^\dagger(\mathrm{D}^\mathrm{b}(X))$ such that $\Psi\Phi(\sigma)=\sigma\cdot\lambda$ [Bri08, Theorem 13.3]. Now, $\Psi\Phi\in\mathrm{Aut}(\mathrm{D}^\mathrm{b}(X),\,\sigma\cdot\mathbb{C})$ and this element induces ϕ. This shows that η is surjective.

It is apparent that $\mathbb{Z}[2]\subseteq\ker(\eta)$. To prove the converse, first pick any $\Phi\in\ker(\eta)$ and let $\lambda\in\mathbb{C}$ be the element which satisfies $\Phi(\sigma)=\sigma\cdot\lambda$. The hypothesis implies that $e^{-i\pi\lambda}=1$, so $\lambda=2m$ for some $m\in\mathbb{Z}$. It follows that $\Phi\circ[-2m]$ fixes σ, and it acts as the identity on $\widetilde{H}(X,\mathbb{Z})$. By [Bri08, Theorem 1.1], we have $\Phi\circ[-2m]=\mathrm{id}$, whence $\Phi=[2m]\in\mathbb{Z}[2]$. This proves that $\ker(\eta)\subseteq\mathbb{Z}[2]$ and thus $\ker(\eta)=\mathbb{Z}[2]$.

We have proved that η is surjective with kernel equal to $\mathbb{Z}[2]$, so it induces the isomorphism in the statement. The finiteness of the groups follows from Lemma 3.3. The proof for the symplectic version is almost the same, so we leave it to the reader. □

Proposition 3.4 implies that an autoequivalence Φ of Gepner type with respect to a stability condition $\sigma\in\mathrm{Stab}^\dagger(X)$ has finite order modulo even shifts. This imposes a strong restriction on possible scalars $\lambda\in\mathbb{C}$ which can appear in the relation $\Phi(\sigma)=\sigma\cdot\lambda$.

Corollary 3.5. *Let X be an algebraic K3 surface and $\Phi\in\mathrm{Aut}(\mathrm{D}^\mathrm{b}(X))$ be of Gepner type with respect to $\sigma\in\mathrm{Stab}^\dagger(X)$. Suppose that Φ has order r modulo even shifts. Then there exists an integer k such that $\Phi(\sigma)=\sigma\cdot\frac{2k}{r}$.*

Proof. Let $Z\in\mathcal{P}^+_0(X)$ be the central charge of σ and $\phi\in\mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),\,Z\cdot\mathbb{C}^*)$ be the isometry induced by Φ. By hypothesis, there exists $\lambda\in\mathbb{C}$ such that $\Phi(\sigma)=\sigma\cdot\lambda$, which implies that $\phi(Z)=Z\cdot e^{-i\pi\lambda}$. Because Φ has order r modulo even shifts, the isometry ϕ has order r. Hence $(e^{-i\pi\lambda})^r=1$. Thus, there exists an integer k such that $\lambda r=2k$, or equivalently, $\lambda=\frac{2k}{r}$. □

The remaining part of this subsection is devoted to the construction of a short exact sequence to be used later. Let us start with the homomorphism

$$\mathrm{Aut}(\mathrm{D}^\mathrm{b}(X),\,\sigma\cdot\mathbb{C})\longrightarrow\mathbb{C} \tag{3.5}$$

which maps Φ to the scalar $\lambda\in\mathbb{C}$ which satisfies $\Phi(\sigma)=\sigma\cdot\lambda$. At the lattice level, there is a similar homomorphism

$$\mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),\,Z\cdot\mathbb{C}^*)\longrightarrow\mathbb{C}^* \tag{3.6}$$

which maps ϕ to the scalar $\kappa\in\mathbb{C}$ which satisfies $\phi(\sigma)=\sigma\cdot\kappa$. These two maps form a commutative diagram

$$\begin{array}{ccc}
\mathrm{Aut}(\mathrm{D}^\mathrm{b}(X),\,\sigma\cdot\mathbb{C}) & \overset{\eta}{\twoheadrightarrow} & \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),\,Z\cdot\mathbb{C}^*) \\
\downarrow{\scriptstyle (3.5)} & & \downarrow{\scriptstyle (3.6)} \\
\mathbb{C} & \xrightarrow{\quad e^{-i\pi(-)}\quad} & \mathbb{C}^*
\end{array}
\qquad\qquad
\begin{array}{ccc}
\Phi & \longmapsto & \phi \\
\downarrow & & \downarrow \\
\lambda & \longmapsto & e^{-i\pi\lambda}
\end{array}$$

where the map η is given in Proposition 3.4. The group $\mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),\, Z\cdot\mathbb{C}^*)$ is finite by Lemma 3.3, so the image of (3.6) is the subgroup $\mu_m \subseteq \mathbb{C}^*$ of m-th roots of unity, where

$$m = m(X,Z) := \text{the cardinality of the image of (3.6)}\ .$$

Notice that m is an even integer since $-\mathrm{id}$ is mapped to $-1 \in \mu_m$. If we consider only μ_m instead of $\mathbb{C}^*$, then the above diagram becomes

$$\begin{array}{ccc} \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X),\, \sigma\cdot\mathbb{C}) & \xrightarrow{\eta} & \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),\, Z\cdot\mathbb{C}^*) \\ \downarrow{\scriptstyle (3.5)} & & \downarrow{\scriptstyle (3.6)} \\ \frac{2}{m}\mathbb{Z} & \xrightarrow{e^{-i\pi(-)}} & \mu_m. \end{array}$$

By involving all the kernels of these maps, we obtained a commutative diagram with exact columns and rows

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X),\, \sigma) & \xrightarrow[\sim]{(3.4)} & \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),\, Z) & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}[2] & \longrightarrow & \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X),\, \sigma\cdot\mathbb{C}) & \xrightarrow{\eta} & \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),\, Z\cdot\mathbb{C}^*) & \longrightarrow & 0 \\ & & \downarrow\wr & & \downarrow{\scriptstyle (3.5)} & & \downarrow{\scriptstyle (3.6)} & & \\ 0 & \longrightarrow & 2\mathbb{Z} & \longrightarrow & \frac{2}{m}\mathbb{Z} & \xrightarrow{e^{-i\pi(-)}} & \mu_m & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0. & & \end{array} \tag{3.7}$$

Here the surjectivity of (3.5) is a consequence of the snake lemma.

The same construction can be applied to the symplectic subgroups to get the same diagram. The only difference is, as $-\mathrm{id}$ is not symplectic, the group μ_m may not contain -1, thus m could possibly be odd.

Proposition 3.6. *Let X be an algebraic K3 surface and $\sigma \in \mathrm{Stab}^\dagger(X)$ be a stability condition. Then there exists an even integer m together with a short exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X),\sigma) & \longrightarrow & \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X),\, \sigma\cdot\mathbb{C})/\mathbb{Z}[2] & \longrightarrow & \mu_m & \longrightarrow & 0 \\ & & & & [1] & \longmapsto & -1 & & \end{array}$$

where $\mu_m \subseteq \mathbb{C}^*$ is the subgroup of m-th roots of unity. The same statement holds with Aut *replaced by* Aut_s *except that* $[1]$ *is not involved and m could possibly be odd.*

Proof. The short exact sequence is obtained by taking the quotient of the middle column by the first column in diagram (3.7). □

3.3 Abelian surfaces and generic twisted K3 surfaces Let X be a complex projective K3 surface, or an abelian surface, equipped with a Brauer class $\alpha \in H^2(X, \mathcal{O}_X^*)$. The pair (X, α) is called *generic* if the bounded derived category $\mathrm{D}^{\mathrm{b}}(X, \alpha)$ of α-twisted coherent sheaves on X does not contain any spherical objects. For instance, all twisted abelian surfaces are generic in this sense. One can define the component $\mathrm{Stab}^\dagger(X, \alpha)$, the domains $\mathcal{P}^+(X, \alpha)$ and $\mathcal{Q}^+(X, \alpha)$, in a similar manner as in Section 3.1; see [HMS08, Section 3] for the details. Note that for a generic twisted (X, α), there is no need to remove the locus orthogonal to (-2)-vectors from its domains as there is no spherical objects.

Due to the absence of spherical objects, the group of autoequivalences and the space of stability conditions of generic twisted surfaces become manageable. Indeed, if X is a generic twisted K3 surface or an arbitrary twisted abelian surface, then the forgetful map

$$\mathrm{Stab}^\dagger(X, \alpha) \longrightarrow \mathcal{P}^+(X, \alpha) : \sigma = (Z, \mathscr{P}) \longmapsto Z$$

is a universal cover with $\mathbb{Z}[2]$ as the group of deck transformations. (See [Bri08, Theorem 15.2] for abelian surfaces, and [HMS08, Theorem 3.15] for generic twisted surfaces.) As a consequence, the map

$$\mathrm{Stab}^\dagger(X, \alpha) \longrightarrow \mathcal{Q}^+(X, \alpha)$$

is a $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-fibration. With this concrete description, let us show that the realization problems have positive answers in these cases.

Proposition 3.7. *Suppose that (X, α) is a generic twisted K3 surface or an arbitrary twisted abelian surface. Then*

- *every finite subgroup of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X, \alpha))/\mathbb{Z}[2]$ fixes a point on $\mathrm{Stab}(X, \alpha)/\mathbb{C}$, and*
- *every finite subgroup of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X, \alpha))$ fixes a point on $\mathrm{Stab}(X)$.*

Proof. The numerical Grothendieck group $\mathcal{N}(X, \alpha)$ has signature $(2, \rho)$ for some positive integer ρ. Let $\mathrm{O}(2, \rho)$ be the orthogonal group of $\mathcal{N}(X, \alpha) \otimes \mathbb{R}$ and $\mathrm{O}^+(2, \rho) \subseteq \mathrm{O}(2, \rho)$ be the subgroup of elements preserving the orientation of a positive 2-plane. Then there is a diffeomorphism

$$\mathcal{Q}^+(X, \alpha) \cong \mathrm{O}^+(2, \rho) / \mathrm{SO}(2) \times \mathrm{O}(\rho).$$

By [HMS08, Theorem 8], we have a homomorphism

$$\eta\colon \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X, \alpha))/\mathbb{Z}[2] \longrightarrow \mathrm{O}^+(2, \rho).$$

Let $G \subseteq \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X, \alpha))/\mathbb{Z}[2]$ be a finite subgroup. Then the image $\eta(G) \subseteq \mathrm{O}^+(2, \rho)$, being finite, is contained in a maximal compact subgroup. All maximal compact subgroups in a Lie group with finitely many connected components are conjugate to each other [HN12, Theorem 14.1.3]. Hence,

$$\eta(G) \subseteq g_0 \left(\mathrm{SO}(2) \times \mathrm{O}(\rho)\right) g_0^{-1} \quad \text{for some} \quad g_0 \in \mathrm{O}^+(2, \rho).$$

Thus $g_0 \left(\mathrm{SO}(2) \times \mathrm{O}(\rho)\right)$ corresponds to a point $[v_0] \in \mathcal{Q}^+(X, \alpha)$ fixed by the action of G.

Let $\sigma \in \mathrm{Stab}^\dagger(X,\alpha)$ be a lift of $[v_0] \in \mathcal{Q}^+(X,\alpha)$ and let $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X,\alpha))$ be a lift of any element of G. Then

$$\Phi(\sigma) = \sigma \cdot g \qquad \text{for some} \qquad g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R}).$$

Let m be a positive integer such that $\Phi^m = [2k]$ for some $k \in \mathbb{Z}$. Then

$$\sigma \cdot 2k = \Phi^m(\sigma) = \sigma \cdot g^m.$$

This implies that g lies in $\mathbb{C} \subseteq \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ and, more precisely, $g = \frac{2k}{m}$. Thus, we conclude that the point $\overline{\sigma} \in \mathrm{Stab}^\dagger(X,\alpha)/\mathbb{C}$ given by σ is fixed by every element of G.

The second statement is a consequence of the first one in view of Lemma 2.3. □

4 The realization problem for generic K3 surfaces

In this section, we compute the explicit structure the group of symplectic autoequivalences modulo even shifts and prove that all its finite subgroups are of Gepner type. This solves the Nielsen realization problem for K3 surfaces of Picard number one in the version of modulo even shifts, which would then imply the ordinary version. After solving the realization problem for both versions, we give a classification of finite subgroups of autoequivalences without modulo even shifts.

4.1 Fricke groups and actions of spherical twists Throughout this section, we let X be a K3 surface of Picard number one with a polarization $h \in \mathrm{NS}(X)$ of degree $h^2 = 2n$. In particular, the numerical Grothendieck group of X is a lattice of signature $(2,1)$ given by

$$\mathcal{N}(X) \cong H^0(X,\mathbb{Z}) \oplus \mathbb{Z}h \oplus H^4(X,\mathbb{Z}).$$

Each element of $\mathcal{N}(X)$ will be denoted as (r,d,s) for some $r,d,s \in \mathbb{Z}$. With this notation, the Mukai pairing between elements of $\mathcal{N}(X)$ has the form

$$(r_1,d_1,s_1)\cdot(r_2,d_2,s_2) = 2n(d_1d_2) - r_1s_2 - r_2s_1.$$

In this setting, there is a holomorphic map from the hyperbolic upper half plane to the period domain $\mathcal{P}^+(X)$ defined as

$$\begin{aligned} \mathcal{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} &\longrightarrow \mathcal{P}^+(X) \\ z &\longmapsto e^{zh} = (1, z, nz^2). \end{aligned}$$

One can verify that this map is injective and its image forms a section of the $\mathrm{GL}^+(2,\mathbb{R})$-fibration $\mathcal{P}^+(X) \longrightarrow \mathcal{Q}^+(X)$, which yields a biholomorphic map

$$\mathcal{H} \xrightarrow{\sim} \mathcal{Q}^+(X) : z \longmapsto [e^{zh}]. \tag{4.1}$$

Each autoequivalence induces an isometry on $\mathcal{N}(X)$ preserving $\mathcal{Q}^+(X)$, which then induces an isometry on $\mathcal{H}$ via the isomorphism above. This defines a homomorphism

$$\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow \mathrm{PSL}(2,\mathbb{R}) \tag{4.2}$$

Based on [Dol96, Theorem 7.1, Remarks 7.2] and [HMS09, Corollary 3], Kawatani gives a description for the image of this homomorphism [Kaw14, Proposition 2.9]: for each positive integer n, the group $\mathrm{PSL}(2,\mathbb{R})$ contains the distinguished element

$$w_n := \begin{pmatrix} 0 & -\frac{1}{\sqrt{n}} \\ \sqrt{n} & 0 \end{pmatrix}, \quad \text{or equivalently,} \quad w_n \colon z \longmapsto -\frac{1}{nz}$$

which is called the *Fricke involution*. Together with the Hecke congruence subgroup

$$\Gamma_0(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{n} \right\},$$

they generate the *Fricke group*

$$\Gamma_0^+(n) := \langle \Gamma_0(n),\ w_n \rangle \subseteq \mathrm{PSL}(2,\mathbb{R}).$$

Notice that $\Gamma_0^+(1) = \Gamma_0(1) = \mathrm{PSL}(2,\mathbb{Z})$.

Example 4.1. Here are some examples of actions of autoequivalences on $\mathcal{H}$. The actions of the tensoring functor $- \otimes \mathcal{O}_X(1)$ and the spherical twist $T_{\mathcal{O}_X}$ on the lattice $\mathcal{N}(X)$ from the left are respectively given by the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ n & 2n & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The matrix for the tensoring functor is computed directly. For the spherical twist, we use the fact that it acts as the reflection along the (-2)-vector $(1,0,1) \in \mathcal{N}(X)$. About their actions on $\mathcal{H}$, let us take the spherical twist for example

$$e^{zh} = (1, z, nz^2) \longmapsto (-nz^2, z, -1) = -nz^2 \left(1, -\tfrac{1}{nz}, \tfrac{1}{nz^2}\right) = -nz^2 e^{-\frac{1}{nz}h}.$$

This shows that it induces the Fricke involution $z \mapsto -\frac{1}{nz}$. One can verify that the tensoring functor induces the translation $z \mapsto z+1$ via a similar computation.

In the following, we give a characterization for the involutions on $\mathcal{H}$ induced by spherical twists.

Lemma 4.2. *Via (4.1), the reflection along a (-2)-vector $\delta = (r,d,s) \in \mathcal{N}(X)$ induces the following involution*

$$\begin{pmatrix} \sqrt{n}d & -\frac{s}{\sqrt{n}} \\ \sqrt{n}r & -\sqrt{n}d \end{pmatrix} = \begin{pmatrix} s & d \\ nd & r \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{\sqrt{n}} \\ \sqrt{n} & 0 \end{pmatrix}$$

which lies in the coset $\Gamma_0(n) w_n \subseteq \Gamma_0^+(n)$. Conversely, every involution in $\Gamma_0(n) w_n$ can be written in the form above, thus is induced by the reflection along a (-2)-vector in $\mathcal{N}(X)$.

Proof. The image of $e^{zh} \in \mathcal{P}^+(X)$ under the reflection equals

$$\begin{aligned} & e^{zh} + (e^{zh} \cdot \delta)\delta \\ & = (1, z, nz^2) + (2ndz - nrz^2 - s)(r, d, s) \\ & = (1 + 2nrdz - nr^2z^2 - rs,\ z + 2nd^2z - nrdz^2 - ds,\ nz^2 + 2ndsz - nrsz^2 - s^2). \end{aligned}$$

We have $\delta^2 = 2nd^2 - 2rs = -2$, or equivalently, $nd^2 - rs = -1$. Applying this to the above expression reduces it to

$$\begin{aligned} & \left(-n(rz-d)^2,\ -(rz-d)(ndz-s),\ -(ndz-s)^2\right) \\ & = -n(rz-d)^2 \left(1,\ \frac{ndz-s}{n(rz-d)},\ n\left(\frac{ndz-s}{n(rz-d)}\right)^2\right) \\ & = -n(rz-d)^2 \exp\left(\frac{ndz-s}{n(rz-d)}h\right) \\ & = -n(rz-d)^2 \exp\left(\frac{\sqrt{n}dz - \frac{s}{\sqrt{n}}}{\sqrt{n}rz - \sqrt{n}d}h\right). \end{aligned}$$

This shows that, on $\mathcal{Q}^+(X)$, the reflection induces the desired transformation. A straightforward computation shows that the transformation decomposes as in the statement, where the first factor belongs to $\Gamma_0(n)$ as $rs - nd^2 = 1$, and the second factor is exactly the Fricke involution w_n. Therefore, the transformation belongs to $\Gamma_0(n)w_n$.

To prove the converse, first note that every element is $\Gamma_0(n)w_n$ can be written as

$$\begin{pmatrix} s & d \\ n\ell & r \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{\sqrt{n}} \\ \sqrt{n} & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{n}d & -\frac{s}{\sqrt{n}} \\ \sqrt{n}r & -\sqrt{n}\ell \end{pmatrix} \quad \text{for some} \quad \begin{pmatrix} s & d \\ n\ell & r \end{pmatrix} \in \Gamma_0(n).$$

Being an involution means that

$$\begin{pmatrix} \sqrt{n}d & -\frac{s}{\sqrt{n}} \\ \sqrt{n}r & -\sqrt{n}\ell \end{pmatrix}^2 = \begin{pmatrix} nd^2 - rs & -s(d-\ell) \\ nr(d-\ell) & n\ell^2 - rs \end{pmatrix} = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let us prove that $d = \ell$. Assume, to the contrary, that $d \neq \ell$, or equivalently, $d - \ell \neq 0$. Hence $r = s = 0$ by the relation above. The same relation then implies $nd^2 = n\ell^2 = \pm 1$, which can happen only if $n = 1$ and $(d, \ell) = \pm(1, -1)$. These data yields the identity element in $\mathrm{PSL}(2, \mathbb{Z})$, which is not considered as an involution. This proves that $d = \ell$. It follows from the first part of the lemma that such an involution can be recovered by the reflection along the (-2)-vector $\delta = (r, d, s) \in \mathcal{N}(X)$. □

Proposition 4.3. *Let X be a K3 surface of Picard number one and degree $2n$. Then the action of a spherical twist on $\mathcal{Q}^+(X) \cong \mathcal{H}$ corresponds to an involution in the coset*

$$\Gamma_0(n)w_n = \begin{cases} \Gamma_0(1) = \mathrm{PSL}(2, \mathbb{Z}) & \text{if} \quad n = 1, \\ \Gamma_0^+(n) \setminus \Gamma_0(n) & \text{if} \quad n \geq 2. \end{cases}$$

Conversely, every involution in this coset is induced by a spherical twist.

Proof. Given a spherical object $\mathcal{E}$, the action of the spherical twist $T_{\mathcal{E}}$ on $\mathcal{N}(X)$ corresponds to the reflection along the (-2)-vector $v(\mathcal{E}) \in \mathcal{N}(X)$. By Lemma 4.2, such a reflection acts on $\mathcal{Q}^+(X) \cong \mathcal{H}$ as an involution in $\Gamma_0(n)w_n$. Conversely, every involution in $\Gamma_0(n)w_n$ is induced by the reflection along a (-2)-vector $\delta \in \mathcal{N}(X)$ by the same lemma. Every (-2)-vector in $\mathcal{N}(X)$ is the Mukai vector of a spherical object. (cf. [Kaw19, Remark 2.10] or [Huy16a, Remark 10.3.3].) Therefore, $\delta = v(\mathcal{E})$ for some spherical object $\mathcal{E}$, and the reflection along δ can be recovered by $T_{\mathcal{E}}$. □

4.2 Fundamental group of the period space The period domain $\mathcal{Q}_0^+(X)$ is constructed from $\mathcal{Q}^+(X)$ by removing any hyperplane section which is orthogonal to a (-2)-vector in $\mathcal{N}(X)$. For a K3 surface of Picard number one, these hyperplane sections are points, which will be called *(-2)-points*. By Lemma 4.2, the isomorphism $\mathcal{H} \cong \mathcal{Q}^+(X)$ given in (4.1) identifies the set of (-2)-points as the set of points on $\mathcal{H}$ fixed by an involution in the coset $\Gamma_0(n)w_n$. In other words,

$$\mathcal{Q}_0^+(X) \cong \mathcal{H} \setminus \{\text{fixed points of involutions in } \Gamma_0(n)w_n\}\,.$$

By a *period space*, we mean the orbifold $\Gamma_0^+(n) \setminus\!\setminus \mathcal{Q}_0^+(X)$ since it parametrizes stability conditions modulo $\mathrm{GL}^+(2,\mathbb{R})$-actions and autoequivalences. Our current task is to derive an explicit formula for its fundamental group.

Let us start with the orbifolds

$$Y_0(n) := \Gamma_0(n) \setminus\!\setminus \mathcal{H} \qquad \text{and} \qquad Y_0^+(n) := \Gamma_0^+(n) \setminus\!\setminus \mathcal{H}.$$

whose underlying topological spaces are classical modular curves with cusps removed. For the curve $Y_0(n)$, let $g = g(n)$ be the genus of its compactification $\overline{Y_0(n)}$ and define

$$\nu_i = \nu_i(n) := \text{the number of elliptic points of orders } i,$$
$$\nu_\infty = \nu_\infty(n) := \text{the number of cusps, namely, the points in } \overline{Y_0(n)} \setminus Y_0(n).$$

It is known that $v_i \neq 0$ only when $i = 2, 3, \infty$, and they can be explicitly computed for each given n using the formulas in [Shi71, Propositions 1.40 & 1.43]. For the curve $Y_0^+(n)$, let us define $g^+ = g^+(n)$ to be the genus of $\overline{Y_0^+(n)}$ and define similarly that

$$\nu_i^+ = \nu_i^+(n) := \text{the number of elliptic points of orders } i,$$
$$\nu_\infty^+ = \nu_\infty^+(n) := \text{the number of cusps, namely, the points in } \overline{Y_0^+(n)} \setminus Y_0^+(n).$$

Furthermore, we will call a non-elliptic point on $Y_0(n)$ or $Y_0^+(n)$ an *ordinary point*.

To attain our goal, the first step is to find formulas for the invariants of $Y_0^+(n)$ in terms of the invariants of $Y_0(n)$. When $n = 1$, the curves $Y_0(1)$ and $Y_0^+(1)$ are isomorphic, so the fact that $\nu_2 = \nu_3 = \nu_\infty = 1$ and $g = 0$ implies that $\nu_2^+ = \nu_3^+ = \nu_\infty^+ = 1$ and $g^+ = 0$ with all the other invariants vanishing. For $n \geq 2$, the fact that $\Gamma_0(n) \subseteq \Gamma_0^+(n)$ is an index 2 subgroup induces the ramified double covering map

$$\overline{Y_0(n)} \longrightarrow \overline{Y_0^+(n)}. \tag{4.3}$$

Ramification points of this map are classified in Fricke's book [Fri28, II, 4, §3]. In the cases that $n = 2, 3, 4$, these points are respectively

($n = 2$) an ordinary point and an elliptic point of order 2,

($n = 3$) an ordinary point and an elliptic point of order 3,

($n = 4$) an ordinary point and a cusp.

To understand the general situation, first define $h(D)$ to be the class number of primitive integral quadratic forms of discriminant D. In general, this number can be computed by Dirichlet's class number formula (cf. [Dav00, Chapter 6]). In our setting, only $D < 0$ needs to be considered, and there are a number of algorithms for computing the class number in this condition (cf. [Coh93, Section 5.3]). Using the class number, let us define

$$\xi = \xi(n) := \begin{cases} h(-4n) & \text{if} \quad n \not\equiv 3 \pmod 4, \\ h(-n) + h(-4n) & \text{if} \quad n \equiv 3 \pmod 4. \end{cases}$$

Then, for each $n \geq 1$, Fricke verified that (4.3) is ramified at ξ many ordinary points and, when $n \geq 5$, only ordinary points occur as ramification points.

Knowing the ramification locus helps us gain a full picture about the distribution of elliptic points and cusps on $Y_0^+(n)$.

Lemma 4.4. *We have $g^+ = 0$ when $n = 1, 2, 3, 4$. The other nonzero invariants in these cases, together with the branch loci of (4.3), are as follows:*

($n = 1$) $\nu_2^+ = \nu_3^+ = \nu_\infty^+ = 1$, *no branch locus in this case.*

($n = 2$) $\nu_2^+ = \nu_4^+ = \nu_\infty^+ = 1$, *where the two elliptic points form the branch locus.*

($n = 3$) $\nu_2^+ = \nu_6^+ = \nu_\infty^+ = 1$, *where the two elliptic points form the branch locus.*

($n = 4$) $\nu_2^+ = 1$, $\nu_\infty^+ = 2$, *where the elliptic point and one cusp form the branch locus.*

For $n \geq 5$, only ν_2^+, ν_3^+, ν_∞^+, g^+ are possibly nonzero, and they are given by the formulas

$$\nu_2^+ = \frac{\nu_2}{2} + \xi, \qquad \nu_3^+ = \frac{\nu_3}{2}, \qquad \nu_\infty^+ = \frac{\nu_\infty}{2}, \qquad 2g^+ = g + 1 - \frac{\xi}{2}.$$

In this case, the branch locus consists of ξ many elliptic points of order 2.

Proof. The statement for $n = 1$ follows from the fact that $Y_0^+(1) \cong Y_0(1)$. In particular, there is no branch locus to discuss in this case.

For $n \geq 2$, the deck transformation of the double cover (4.3) maps an ordinary point to an ordinary point, an elliptic point to an elliptic point of the same order, and a cusp to a cusp. Away from the ramification locus, (4.3) maps a pair of points to a point of the same type. Along the ramification locus, it maps an ordinary point to an elliptic point of order 2, an elliptic point of order i to an elliptic point of order $2i$, and a cusp to a cusp.

For $n = 2, 3, 4$, map (4.3) is ramified at exactly two points, so the fact that $g = 0$ and the Riemann–Hurwitz formula give $g^+ = 0$. As for the other invariants and base loci:

- The curve $Y_0(2)$ (resp. $Y_0(3)$) has an elliptic point of order 2 (resp. order 3) and two cusps. The map (4.3) is ramified at an ordinary point, the elliptic point, and is unramified at the two cusps. This gives $Y_0^+(2)$ (resp. $Y_0^+(3)$) an elliptic point of order 2, an elliptic point of order 4 (resp. order 6), and a cusp. Hence $\nu_2^+ = 1$, $\nu_4^+ = 1$ (resp. $\nu_6^+ = 1$), $\nu_\infty^+ = 1$, and the two elliptic points form the branch locus.

- The curve $Y_0(4)$ has three cusps. The map (4.3) is ramified at an ordinary point, one of the three cusps, and exchanges the remaining two cusps. This produces an elliptic point of order 2 and two cusps on $Y_0^+(4)$. Hence $\nu_2^+ = 1$ and $\nu_\infty^+ = 2$. In this case, the elliptic point and one of the two cusps form the branch locus.

When $n \geq 5$, the ramification locus of (4.3) consists of ξ many ordinary points, so the map produces ξ many new elliptic points of order 2 and, for each type of existing elliptic points or cusps, it reduces their quantity to half. This, together with the Riemann–Hurwitz formula, gives the formulas in the statement. In this case, the ξ many new elliptic points form the branch locus. □

Recall that a (-2)-point on $\mathcal{H}$ is a point fixed by an involution in $\Gamma_0(n)w_n$. The image of such a point on the quotient $Y_0^+(n) = \Gamma_0^+(n) \setminus\setminus \mathcal{H}$ will also be called a (-2)-point. Let us give a precise description about how they distribute on $Y_0^+(n)$.

Lemma 4.5. *The only (-2)-point on $Y_0^+(1)$ is the elliptic point of order 2. For $n \geq 2$, the set of (-2)-points on $Y_0^+(n)$ coincides with the set of the branch points of (4.3) whose preimage is an ordinary point or an elliptic point of odd order. Thus it consists of*

- *the only elliptic point of order 2 when $n = 2, 4$,*
- *the two elliptic points, one of order 2 and the other of order 6, when $n = 3$,*
- *ξ-many elliptic points of order 2 when $n \geq 5$.*

Proof. For $n = 1$, we have $\Gamma_0^+(1) = \Gamma_0(1)$, so the set of (-2)-points on $\mathcal{H}$ is the same as the set of elliptic points of order 2, which forms a single point on $Y_0^+(1) = \Gamma_0^+(1) \setminus\setminus \mathcal{H}$.

Suppose that $n \geq 2$ and let $p \in \mathcal{H}$ be a (-2)-point. By hypothesis, there exists an involution $w \in \Gamma_0^+(n) \setminus \Gamma_0(n)$ which fixes p. The involution w represents the deck transformation of (4.3), so p corresponds to a branch point of this double cover. Suppose that, as a point on $Y_0^+(n)$, the preimage of p in $Y_0(n)$ is an elliptic point of even order, or equivalently, its stabilizer $\Gamma_0(n)_p \subseteq \Gamma_0(n)$ is a cyclic subgroup of even order. Being of even order implies that $\Gamma_0(n)_p$ contains an involution g. Now, both w and g are elliptic elements of order 2 fixing the same point p, which can happen only if $w = g \in \Gamma_0(n)$, contradiction. This shows that every (-2)-point on $Y_0^+(n)$ is contained in the set of branch points whose preimage is an ordinary point or an elliptic point of odd order.

Conversely, let $p \in \mathcal{H}$ be a point whose image in $Y_0(n)$ is an elliptic point of odd order and belongs to the ramification locus of (4.3). This implies that the stabilizer $\Gamma_0(n)_p \subseteq \Gamma_0(n)$ is cyclic of odd order m and the stabilizer $\Gamma_0^+(n)_p \subseteq \Gamma_0^+(n)$ is cyclic of order $2m$. Given such a condition, one can find an involution $w \in \Gamma_0^+(n)_p \setminus \Gamma_0(n)_p$, which implies that p is a (-2)-point. This shows that every branch point whose preimage is an ordinary point or an elliptic point of odd order is a (-2)-point.

Observe that, for a branch point $p \in Y_0^+(n)$, if its preimage in $Y_0(n)$ is an ordinary point or an elliptic point of odd order, then p itself is an elliptic point of order $2m$ for some odd m, and vice versa. This observation, together with Lemma 4.4 and what we proved above, gives us the precise description about the set of (-2)-points in each case. □

We are now ready to compute the fundamental group of $\Gamma_0^+(n) \setminus\setminus \mathcal{Q}_0^+(X)$. Notice that this space contains a cusp given by $i\infty$ for every n. A counterclockwise loop around this cusp corresponds to the element $z \mapsto z+1$ in $\Gamma_0^+(n)$ induced by the functor $- \otimes \mathcal{O}_X(1)$. In the following, we will call a cusp *not* given by $i\infty$ a *real cusp*.

Proposition 4.6. *Let X be a K3 surface of Picard number one and degree $2n$. Then the fundamental group of $\Gamma_0^+(n) \setminus\setminus \mathcal{Q}_0^+(X)$ can be expressed as*

$$\pi_1^{\mathrm{orb}}(\Gamma_0^+(n) \setminus\setminus \mathcal{Q}_0^+(X)) \cong \begin{cases} \mathring{\mathbb{Z}} * \mathbb{Z}_3 & \text{if} \quad n = 1 \\ \mathring{\mathbb{Z}} * \mathbb{Z}_4 & \text{if} \quad n = 2 \\ \mathring{\mathbb{Z}} * \mathring{\mathbb{Z}} & \text{if} \quad n = 3 \\ \mathring{\mathbb{Z}} * \check{\mathbb{Z}} & \text{if} \quad n = 4 \\ \mathbb{Z}_2^{*\frac{\nu_2}{2}} * \mathbb{Z}_3^{*\frac{\nu_3}{2}} * \mathring{\mathbb{Z}}^{*\xi} * \check{\mathbb{Z}}^{*\left(\frac{\nu_\infty}{2}-1\right)} * \mathbb{Z}^{*\left(g+1-\frac{\xi}{2}\right)} & \text{if} \quad n \geq 5 \end{cases}$$

where a copy of $\mathbb{Z}$ is decorated as $\mathring{\mathbb{Z}}$ (resp. $\check{\mathbb{Z}}$) if and only if it is generated by a loop around a (-2)-point (resp. a real cusp).

Proof. For $n = 1, 2, 3, 4$, Lemmas 4.4 and 4.5 give us the following information about elliptic points and holes on $\Gamma_0^+(n) \setminus\setminus \mathcal{Q}_0^+(X)$:

- When $n = 1, 2$, there is an elliptic point of order 3, respectively, of order 4. In both cases, the underlying topological space is a sphere with two holes, where one of them is a (-2)-point and the other is a cusp given by $i\infty$.
- When $n = 3, 4$, there is no elliptic point. In both cases, the underlying topological space is a sphere with three holes. When $n = 3$, two holds are (-2)-points while the third hole is a cusp given by $i\infty$. When $n = 4$, one hole is a (-2)-point, the other two holes are cusps, and one of the cusps is given by $i\infty$.

In each of these cases, the explicit formula for the fundamental group can be derived from the Seifert–Van Kampen theorem for orbifolds (cf. [Car22, Theorem 2.2.3]).

Suppose that $n \geq 5$ from now on. To derive the formula in this case, let us first think of the space $\Gamma_0^+(n) \setminus\setminus \mathcal{Q}_0^+(X)$ as a union $D \cup S$ where

- D is a disk containing all elliptic points and holes,
- S is a connected oriented surface of genus g^+ with a hole,

such that $D \cap S$ is an annulus. Then the Seifert–Van Kampen theorem gives

$$\pi_1^{\mathrm{orb}}(\Gamma_0^+(n) \setminus\setminus \mathcal{Q}_0^+(X)) \;\cong\; \pi_1^{\mathrm{orb}}(D) *_{\pi_1(D\cap S)} \pi_1(S). \tag{4.4}$$

Notice that $\pi_1(S) \cong \mathbb{Z}^{*2g^+}$. By Lemmas 4.4 and 4.5, the disk D contains $\frac{\nu_2}{2}$ (resp. $\frac{\nu_3}{2}$) many elliptic points of order 2 (resp. order 3). It also has $\xi + \frac{\nu_\infty}{2}$ many holes, where ξ many of them are (-2)-points and the others are cusps. This implies that

$$\pi_1^{\mathrm{orb}}(D) \cong \mathbb{Z}_2^{*\frac{\nu_2}{2}} * \mathbb{Z}_3^{*\frac{\nu_3}{2}} * \mathring{\mathbb{Z}}^{*\xi} * \check{\mathbb{Z}}^{*\left(\frac{\nu_\infty}{2}-1\right)} * \mathbb{Z} \tag{4.5}$$

where the last copy of $\mathbb{Z}$ is generated by a loop around the cusp given by $i\infty$.

Let $\{\gamma_1, \ldots, \gamma_N\}$, where $N = \xi + \frac{1}{2}(\nu_2 + \nu_3 + \nu_\infty)$, be a basis for (4.5) such that each γ_i is represented by a counterclockwise loop around an elliptic point or a hole. Upon reordering, we can assume that γ_N corresponds to the cusp at $i\infty$. In this setting, $\pi_1(D \cap S) \cong \mathbb{Z}$ is generated by

$$\gamma_D := \prod_{i=1}^{N} \gamma_i.$$

Because γ_N is of infinite order, if we replace γ_N by γ_D, then the set $\{\gamma_1, \ldots, \gamma_{N-1}, \gamma_D\}$ still forms a basis for (4.5). By writing $\pi_1^{\mathrm{orb}}(D)$ as a free product in terms of this basis, the factor generated by γ_D will be absorbed into $\pi_1(S)$ in the amalgamated product (4.4). Thus,

$$\begin{aligned}\pi_1^{\mathrm{orb}}(\Gamma_0^+(n) \setminus\setminus \mathcal{Q}_0^+(X)) &\cong \left(\mathbb{Z}_2^{*\frac{\nu_2}{2}} * \mathbb{Z}_3^{*\frac{\nu_3}{2}} * \mathring{\mathbb{Z}}^{*\xi} * \check{\mathbb{Z}}^{*\left(\frac{\nu_\infty}{2}-1\right)} * \mathbb{Z}\right) *_{\mathbb{Z}} \mathbb{Z}^{*2g^+} \\ &\cong \mathbb{Z}_2^{*\frac{\nu_2}{2}} * \mathbb{Z}_3^{*\frac{\nu_3}{2}} * \mathring{\mathbb{Z}}^{*\xi} * \check{\mathbb{Z}}^{*\left(\frac{\nu_\infty}{2}-1\right)} * \mathbb{Z}^{*2g^+}.\end{aligned}$$

The desired expression then follows from the fact that $2g^+ = g + 1 - \frac{\xi}{2}$. □

Remark 4.7. The map $\mathcal{Q}^+(X) \longrightarrow \Gamma_0^+(n) \setminus\setminus \mathcal{Q}^+(X)$ is a universal cover with $\Gamma_0^+(n)$ as the group of deck transformations. Hence $\Gamma_0^+(n)$ is isomorphic to $\pi_1^{\mathrm{orb}}(\Gamma_0^+(n) \setminus\setminus \mathcal{Q}^+(X))$ (cf. [Car22, Proposition 2.3.5 (i)]). Following the same argument as in the proof of Proposition 4.6, one can deduce that

$$\Gamma_0^+(n) \cong \pi_1^{\mathrm{orb}}(\Gamma_0^+(n) \setminus\setminus \mathcal{Q}^+(X)) \cong \begin{cases} \mathbb{Z}_2 * \mathbb{Z}_3 & \text{if} \quad n = 1 \\ \mathbb{Z}_2 * \mathbb{Z}_4 & \text{if} \quad n = 2 \\ \mathbb{Z}_2 * \mathbb{Z}_6 & \text{if} \quad n = 3 \\ \mathbb{Z}_2 * \mathbb{Z} & \text{if} \quad n = 4 \\ \mathbb{Z}_2^{*\left(\frac{\nu_2}{2}+\xi\right)} * \mathbb{Z}_3^{*\frac{\nu_3}{2}} * \mathbb{Z}^{*\left(g+\frac{\nu_\infty-\xi}{2}\right)} & \text{if} \quad n \geq 5. \end{cases}$$

The inclusion $\mathcal{Q}_0^+(X) \subseteq \mathcal{Q}^+(X)$ induces a surjective homomorphism

$$\pi_1^{\mathrm{orb}}(\Gamma_0^+(n) \setminus\setminus \mathcal{Q}_0^+(X)) \twoheadrightarrow \pi_1^{\mathrm{orb}}(\Gamma_0^+(n) \setminus\setminus \mathcal{Q}^+(X))$$

which transforms a $\mathring{\mathbb{Z}}$ factor into a $\mathbb{Z}_2$ factor and leaves all the other factors invariant. Its kernel is the free product of all the subgroups $2\mathring{\mathbb{Z}} \subseteq \mathring{\mathbb{Z}}$ and their conjugates, which is canonically isomorphic to $\pi_1(\mathcal{Q}_0^+(X))$.

4.3 Finite subgroups modulo even shifts and realization For a K3 surface X of Picard number one and degree $2n$, the space $\mathrm{Stab}^\dagger(X)$ is simply connected and invariant under the actions of autoequivalences [BB17, Theorem 1.3], which gives us a universal cover

$$\mathrm{Stab}^\dagger(X)/\widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \longrightarrow \Gamma_0^+(n) \setminus\setminus \mathcal{Q}_0^+(X). \tag{4.6}$$

When $n \geq 2$, the group $\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ is isomorphic to $\pi_1^{\mathrm{orb}}(\Gamma_0^+(n) \setminus\setminus \mathcal{Q}_0^+(X))$, where the latter the group of deck transformations of the above cover [BB17, Remark 7.2]. Proposition 4.6 then implies that

$$\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2] \cong \begin{cases} \mathbb{Z} * \mathbb{Z}_4 & \text{if} \quad n = 2 \\ \mathbb{Z} * \mathbb{Z} & \text{if} \quad n = 3, 4 \\ \mathbb{Z}_2^{*\frac{\nu_2}{2}} * \mathbb{Z}_3^{*\frac{\nu_3}{2}} * \mathbb{Z}^{*\left(g + \frac{\nu_\infty + \xi}{2}\right)} & \text{if} \quad n \geq 5. \end{cases} \tag{4.7}$$

The case $n = 1$ needs a special treatment. In this case, the covering involution of the double cover $X \to \mathbb{P}^2$ induces an anti-symplectic autoequivalence

$$\iota \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$$

which lives in the center [BK22, Example 8.4 (i)] and acts trivially on $\mathrm{Stab}^\dagger(X)$ [Huy12, Lemma A.3]. The composition $\iota[1]$ is symplectic with square equal to $[2]$. Combining these facts, we conclude that $\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}(\iota[1])$ is isomorphic to $\pi_1^{\mathrm{orb}}(\Gamma_0^+(n) \setminus\setminus \mathcal{Q}_0^+(X))$. It then follows from Proposition 4.6 that

$$\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}(\iota[1]) \cong \mathbb{Z} * \mathbb{Z}_3. \tag{4.8}$$

This formula, together with (4.7), allows us to classify finite subgroups of symplectic autoequivalences modulo $\mathbb{Z}[2]$ up to conjugation.

Lemma 4.8. *Let X be a K3 surface of Picard number one and degree $2n$. Then, for every maximal finite subgroup $G_s \subseteq \mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$, it holds that*

$$G_s \cong \begin{cases} \mathbb{Z}_6 & \text{if} \quad n = 1 \\ \mathbb{Z}_4 & \text{if} \quad n = 2 \\ 0 & \text{if} \quad n = 3, 4 \\ 0,\ \mathbb{Z}_2,\ \mathbb{Z}_3 & \text{if} \quad n \geq 5. \end{cases}$$

- *When $n = 1, 2$, there exists one and only one such subgroup up to conjugation.*
- *When $n \geq 5$, there exist $\frac{\nu_2}{2}$ (resp. $\frac{\nu_3}{2}$) many such subgroups isomorphic to $\mathbb{Z}_2$ (resp. $\mathbb{Z}_3$) up to conjugation.*

Moreover, every such subgroup, if nontrivial, fixes one and only one point on $\mathcal{Q}_0^+(X)$.

Proof. Suppose that $n \geq 2$. As a consequence of Kurosh's theorem (cf. [Ser03, Chapter I, Section 5.5]), the subgroup G_s, if nontrivial, is conjugate to one of the finite cyclic factors in the free product (4.7). This gives the classification up to isomorphism and their numbers up to conjugation for $n \geq 2$.

Now assume that $n = 1$. In this case, we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}_2(\iota[1]) \longrightarrow \mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2] \xrightarrow{f} \mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}(\iota[1]) \longrightarrow 0.$$

Let $C_3 \cong \mathbb{Z}_3$ be the finite cyclic factor in the free product (4.8). Then its preimage under f fits into the short exact sequence

$$0 \longrightarrow \mathbb{Z}_2(\iota[1]) \longrightarrow C_6 := f^{-1}(C_3) \xrightarrow{f} C_3 \longrightarrow 0.$$

The group C_6 has order 6, so it is isomorphic to either $\mathbb{Z}_6$ or the symmetric group S_3. It follows that $C_6 \cong \mathbb{Z}_6$ because S_3 has a trivial center. According to Kurosh's theorem, every nontrivial finite subgroup of $\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}(\iota[1])$ is conjugate to C_3. Therefore, we are able to find $\alpha \in \mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ such that

$$f(\alpha) \cdot f(G_s) \cdot f(\alpha)^{-1} \subseteq C_3 \qquad \text{whence} \qquad \alpha \cdot G_s \cdot \alpha^{-1} \subseteq C_6.$$

Because G_s is maximal, the latter inclusion is an equality. This proves that every maximal finite subgroup of $\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ is isomorphic to $\mathbb{Z}_6$ and is unique up to conjugation for the case $n = 1$.

Every nontrivial finite $G_s \subseteq \mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ has a unique fixed point on $\mathcal{Q}_0^+(X)$ as its image in $\Gamma_0^+(n)$ is generated by an elliptic element with this property. □

Example 4.9. Canonaco–Karp [CK08] proved that, if X is a smooth hypersurface in the weighted projective space $\mathbb{P}(w_0, \ldots, w_m)$, then the composition of autoequivalences

$$\Theta := (- \otimes \mathcal{O}_X(1)) \circ T_{\mathcal{O}_X}$$

satisfies the relation $\Theta^w = [2]$ where $w := \sum_{i=0}^m w_i$. If X is a K3 surface of Picard number one and degree 2 or 4, this relation gives:

- If X has degree 2, then it can be realized as a hypersurface in the weighted projective space $\mathbb{P}(3, 1, 1, 1)$, so we get $\Theta^6 = [2]$ in this case.
- If X has degree 4, that is, a quartic hypersurface in $\mathbb{P}^3$, then we obtain $\Theta^4 = [2]$.

In both cases, every maximal finite subgroup of $\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ is generated by Θ up to conjugation by Lemma 4.8. In the degree 2 case, one can derive the equation

$$\Theta^3 \equiv \iota[1] \pmod{\mathbb{Z}[2]}$$

using the same lemma and the fact that the autoequivalences on both sides are symplectic and involutive modulo $\mathbb{Z}[2]$.

According to Lemma 4.8, all subgroups of symplectic autoequivalences which are finite modulo even shifts are cyclic. The realization problem for symplectic autoequivalences will be solved based on this observation and the following lemma.

Lemma 4.10. *Let X be a K3 surface of Picard number one and $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ be an element of finite order. If there exists $\Psi \in \mathcal{I}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ such that $\Phi\Psi$ or $\Psi\Phi$ fixes a point on $\mathrm{Stab}^\dagger(X)/\mathbb{C}$, then Φ fixes a point on $\mathrm{Stab}^\dagger(X)/\mathbb{C}$.*

Proof. Assume that $\Phi\Psi$ fixes a point on $\mathrm{Stab}^\dagger(X)/\mathbb{C}$. Then $\Phi\Psi$ is of finite order by Proposition 3.4. Thus, there exists an integer $m > 0$ such that $\Phi^m = (\Phi\Psi)^m = 1$. By [BB17, Theorem 1.4], the group $\mathcal{I}(\mathrm{D^b}(X))/\mathbb{Z}[2]$ is freely generated by squares of spherical twists, so we can write

$$\Psi = T_{S_1}^{2k_1} \cdots T_{S_\ell}^{2k_\ell}$$

where $S_1, \ldots, S_\ell$ are spherical objects with $T_{S_i} \neq T_{S_{i+1}}$ for all i. Then

$$\begin{aligned} 1 = (\Phi\Psi)^m &= (\Phi\Psi\Phi^{-1})(\Phi^2\Psi\Phi^{-2}) \cdots (\Phi^m\Psi\Phi^{-m})\Phi^m \\ &= (\Phi\Psi\Phi^{-1})(\Phi^2\Psi\Phi^{-2}) \cdots (\Phi^m\Psi\Phi^{-m}) \\ &= \left(T_{\Phi(S_1)}^{2k_1} \cdots T_{\Phi(S_\ell)}^{2k_\ell}\right)\left(T_{\Phi^2(S_1)}^{2k_1} \cdots T_{\Phi^2(S_\ell)}^{2k_\ell}\right) \cdots \left(T_{\Phi^m(S_1)}^{2k_1} \cdots T_{\Phi^m(S_\ell)}^{2k_\ell}\right). \end{aligned}$$

There is no nontrivial relation among squares of spherical twists in our setting. In order for cancellations to occur in the last expression, ℓ has to be even, and we must have

$$k_{\ell-i+1} = -k_i, \qquad S_{\ell-i+1} = \Phi(S_i) \qquad \text{for} \qquad 1 \leq i \leq \frac{\ell}{2}.$$

If we write $p = \frac{\ell}{2}$, then these relations turn Ψ into the form

$$\Psi = T_{S_1}^{2k_1} \cdots T_{S_p}^{2k_p} T_{\Phi(S_p)}^{-2k_p} \cdots T_{\Phi(S_1)}^{-2k_1}$$

which can be rewritten as $\Psi = \Theta\Phi\Theta^{-1}\Phi^{-1}$ with $\Theta = T_{S_1}^{2k_1} \cdots T_{S_p}^{2k_p}$. By hypothesis, there exists $\overline{\sigma} \in \mathrm{Stab}^\dagger(X)/\mathbb{C}$ fixed by $\Phi\Psi$. Now, we have

$$\Phi\Theta\Phi\Theta^{-1}\Phi^{-1}(\overline{\sigma}) = \overline{\sigma}, \qquad \text{or equivalently,} \qquad \Phi(\Theta^{-1}\Phi^{-1}\overline{\sigma}) = \Theta^{-1}\Phi^{-1}\overline{\sigma}.$$

Notice that $\Theta^{-1}\Phi^{-1}\overline{\sigma} \in \mathrm{Stab}^\dagger(X)/\mathbb{C}$ since $\mathrm{Stab}^\dagger(X)$ is invariant under the action of autoequivalences [BB17, Theorem 1.3]. This proves the statement under the assumption that $\Phi\Psi$ has a fixed point.

Now assume that $\Psi\Phi$ fixes a point $\overline{\sigma} \in \mathrm{Stab}^\dagger(X)/\mathbb{C}$. Then its inverse $\Phi^{-1}\Psi^{-1}$ fixes $\overline{\sigma}$ as well. This implies that Φ^{-1} fixes a point on $\mathrm{Stab}^\dagger(X)/\mathbb{C}$ by what we have proved, which then implies that Φ fixes a point on $\mathrm{Stab}^\dagger(X)/\mathbb{C}$. □

Proposition 4.11. *Suppose that X is a K3 surface of Picard number one. Then*

- *every finite subgroup of $\mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2]$ fixes a point on $\mathrm{Stab}^\dagger(X)/\mathbb{C}$, and*
- *every finite subgroup of $\mathrm{Aut}_s(\mathrm{D^b}(X))$ fixes a point on $\mathrm{Stab}^\dagger(X)$.*

Proof. By Lemma 4.8, every finite subgroup of $\mathrm{Aut}_s(\mathrm{D^b}(X))/\mathbb{Z}[2]$ is cyclic, so it suffices to prove that, if $\Phi \in \mathrm{Aut}_s(\mathrm{D^b}(X))$ is of finite order modulo $\mathbb{Z}[2]$, then its action on $\mathrm{Stab}^\dagger(X)/\mathbb{C}$ has a fixed point. The same lemma also asserts that Φ fixes a point on $\mathcal{Q}_0^+(X)$. Hence, if we let ϕ be the Hodge isometry induced by Φ, then there exists $Z \in \mathcal{P}_0^+(X)$ such that $\phi(Z) = Z \cdot \kappa$ for some $\kappa \in \mathrm{GL}^+(2, \mathbb{R})$. In fact, we have $\kappa \in \mathbb{C}^*$ because ϕ is of finite order.

Let $\sigma \in \mathrm{Stab}^\dagger(X)$ be any stability condition with central charge Z and let $\lambda \in \mathbb{C}$ be any element which satisfies $e^{-i\pi\lambda} = \kappa$. Then the central charges of both $\Phi(\sigma)$ and $\sigma \cdot \lambda$ are equal to $Z \cdot \kappa$. By [Bri08, Theorem 1.1], there exists $\Psi \in \mathcal{I}(\mathrm{D^b}(X))$ such that $\Psi\Phi(\sigma) = \sigma \cdot \lambda$, so $\Psi\Phi$ fixes a point on $\mathrm{Stab}^\dagger(X)/\mathbb{C}$, thus Φ has a fixed point as well by Lemma 4.10.

The second statement is a consequence of the first one by Lemma 2.3. □

In order to classify finite subgroups of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ and solve the realization problem for K3 surfaces of Picard number one, we need the following lemma.

Lemma 4.12. *Let X be a K3 surface of odd Picard number. Then*

$$\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2] \;\cong\; (\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]) \times \mathbb{Z}_2[1].$$

In particular, given a subgroup $G \subseteq \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ and its subgroup $G_s \subseteq G$ of symplectic elements, G is either identical to G_s or isomorphic to $G_s \times \mathbb{Z}_2[1]$.

Proof. Let $T(X)$ be the transcendental lattice of X. Then our hypothesis implies that the only Hodge isometries on $T(X)$ are ± 1 [Huy16a, Corollary 3.3.5]. Then the restriction of an autoequivalence to its action on $T(X)$ induces the short exact sequence

$$0 \longrightarrow \mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2] \longrightarrow \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2] \xrightarrow{\tau} \mathbb{Z}_2 \longrightarrow 0.$$

The surjection τ has section induced by $[1]$, so the middle term splits as a semidirect product

$$\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2] \;\cong\; (\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]) \rtimes \mathbb{Z}_2[1]$$

with $[1]$ acting on $\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ by conjugation. Because $[1]$ lives in the center, this semidirect product is actually a direct product. □

Proof of Theorem 1.2. Because G is maximal, it contains $[1]$. Thus $G \cong G_s \times \mathbb{Z}_2[1]$ by Lemma 4.12. The remaining part of the statement follows from Lemma 4.8. □

The following theorem gives an affirmative answer to the realization problem for K3 surfaces of Picard number one.

Theorem 4.13. *Let X be a K3 surface of Picard number one. Then*

- *every finite subgroup of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ fixes a point on $\mathrm{Stab}^{\dagger}(X)/\mathbb{C}$, and*
- *every finite subgroup of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ fixes a point on $\mathrm{Stab}^{\dagger}(X)$.*

Proof. Let $G \subseteq \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ be a maximal finite subgroup. By Lemma 4.12, it holds that $G \cong G_s \times \mathbb{Z}_2[1]$ where $G_s \subseteq G$ is the symplectic subgroup. Notice that $[1]$ fixes every point on $\mathrm{Stab}^{\dagger}(X)/\mathbb{C}$. By Proposition 4.11, G_s fixes a point on $\mathrm{Stab}^{\dagger}(X)/\mathbb{C}$, which implies that G has a fixed point as well, thus proves the first statement. The second statement then follows from Lemma 2.3. □

Proof of Theorem 1.4. Theorem 4.13 and Proposition 3.4 together give the statement. □

Corollary 4.14. *Let X be a K3 surface of Picard number one and $G \subseteq \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ be a maximal finite subgroup. Then there exists $\sigma \in \mathrm{Stab}^{\dagger}(X)$ such that*

$$G = \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X), \sigma).$$

The same statement holds with Aut *replaced by* Aut_s.

Proof. Theorem 4.13 implies that $G \subseteq \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X), \sigma)$ for some $\sigma \in \mathrm{Stab}^{\dagger}(X)$. The inclusion is an equality since G is a maximal finite subgroup and $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X), \sigma)$ is finite [Huy16b, Remark 1.2]. The proof for the symplectic version is the same. □

Corollary 4.15. *Let X be a K3 surface of Picard number one and $G \subseteq \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ be a subgroup containing $[2]$ such that the image $\overline{G} \subseteq \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ is a maximal finite subgroup. Then there exists $\sigma \in \mathrm{Stab}^{\dagger}(X)$ such that*

$$G = \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X),\, \sigma \cdot \mathbb{C}).$$

The same statement holds with Aut *replaced by* Aut_s.

Proof. Theorem 4.13 asserts that there exists $\sigma \in \mathrm{Stab}^{\dagger}(X)$ such that

$$G \subseteq \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X),\, \sigma \cdot \mathbb{C}). \tag{4.9}$$

Modulo even shifts, (4.9) gives the inclusion

$$\overline{G} = G/\mathbb{Z}[2] \subseteq \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X),\, \sigma \cdot \mathbb{C})/\mathbb{Z}[2]. \tag{4.10}$$

By Proposition 3.4, the group on the right hand side is finite, so this inclusion is an equality since $\overline{G}$ is maximal. For every $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X),\, \sigma \cdot \mathbb{C})$, the fact that (4.10) is an equality implies that $\Phi[m] \in G$ for some even m. Because G contains $[2]$, we have $\Phi \in G$. Hence (4.9) is an equality. The proof for the symplectic version is the same. □

5 Finite subgroups, distribution, and further classification

In this section, we continue our discussion in the case of generic K3 surfaces based on the affirmative answer to the Nielsen realization problem. We will give a classification of finite subgroups of autoequivalences up to conjugation with counting formulas for the numbers of conjugacy classes. Then we will give an explicit description of the locus in the space of stability conditions which parametrizes Gepner type points. In the last part of this section, we discuss a classification of autoequivalences in terms of their actions on the hyperbolic period domain, and how different notions of categorical entropy could enter the picture.

5.1 Finite subgroups of autoequivalences and involutions

The classification of subgroups which are finite modulo even shifts is done in Theorem 1.2. The purpose of this part is to classify finite subgroups without modulo even shifts. As shown in Corollary 4.14, every such subgroup is contained in $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X), \sigma)$ for some $\sigma \in \mathrm{Stab}^{\dagger}(X)$. This reduces the problem to the classification of subgroups in the latter form.

Lemma 5.1. *Let X be a K3 surface of Picard number one and $\sigma \in \mathrm{Stab}^{\dagger}(X)$ be a stability condition. Then every $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X), \sigma)$ satisfies $\Phi^2 = \mathrm{id}$.*

Proof. Let $Z \in \mathcal{P}_0^+(X)$ be the central charge of σ. Then the group $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X), \sigma)$ is isomorphic to $\mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}), Z)$ by [Huy16b, Proposition 1.4]. Therefore, it suffices to prove that every $\phi \in \mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}), Z)$ satisfies $\phi^2 = \mathrm{id}$. Let $P \subseteq \mathcal{N}(X) \otimes \mathbb{R}$ be the positive plane spanned by Z. Then the actions of ϕ on $\mathcal{N}(X)$ and $T(X)$ are as follows:

- ϕ acts on $\mathcal{N}(X)$ either trivially or as the reflection across P since these are the only elements in $\mathrm{O}(\mathcal{N}(X)\otimes\mathbb{R})$ fixing P pointwisely.
- ϕ acts on $T(X)$ as $\pm\mathrm{id}$ by [Huy16a, Corollary 3.3.5].

It follows that ϕ^2 acts trivially on both $\mathcal{N}(X)$ and $T(X)$, which implies that $\phi^2=\mathrm{id}$. □

Lemma 5.2. *Let X be a K3 surface of Picard number one and $\sigma\in\mathrm{Stab}^\dagger(X)$ be a stability condition. Then the group $\mathrm{Aut}(\mathrm{D}^\mathrm{b}(X),\sigma)$ is either trivial or isomorphic to $\mathbb{Z}_2$.*

Proof. Consider the short exact sequence given by Proposition 3.6:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Aut}(\mathrm{D}^\mathrm{b}(X),\sigma) & \longrightarrow & \mathrm{Aut}(\mathrm{D}^\mathrm{b}(X),\ \sigma\cdot\mathbb{C})/\mathbb{Z}[2] & \longrightarrow & \mu_m & \longrightarrow & 0 \\ & & & & & & [1] & \longmapsto & -1 \end{array} \tag{5.1}$$

where $\mu_m\subseteq\mathbb{C}^*$ is the group of m-th roots of unity. By Lemma 4.12, the group in the middle splits as:

$$\mathrm{Aut}(\mathrm{D}^\mathrm{b}(X),\ \sigma\cdot\mathbb{C})/\mathbb{Z}[2]\ \cong\ (\mathrm{Aut}_s(\mathrm{D}^\mathrm{b}(X),\ \sigma\cdot\mathbb{C})/\mathbb{Z}[2])\times\mathbb{Z}_2[1]. \tag{5.2}$$

The sequence (5.1) realizes $\mathrm{Aut}(\mathrm{D}^\mathrm{b}(X),\sigma)$ as a subgroup of (5.2). We have already known that $\mathrm{Aut}(\mathrm{D}^\mathrm{b}(X),\sigma)$ contains only the identity or involutions, which suggests us to classify involutive elements in (5.2). We have an inclusion

$$\mathrm{Aut}_s(\mathrm{D}^\mathrm{b}(X),\ \sigma\cdot\mathbb{C})/\mathbb{Z}[2]\ \subseteq\ \mathrm{Aut}_s(\mathrm{D}^\mathrm{b}(X))/\mathbb{Z}[2]$$

where the left hand side is finite by Proposition 3.4. Since all finite subgroups of the right hand side are cyclic by Lemma 4.8, the left hand side is cyclic, thus it contains either none or only one involution. This implies that the only subgroup of (5.2) which consists of involutions is either $0\times\mathbb{Z}_2[1]$ or $\mathbb{Z}_2\times\mathbb{Z}_2[1]$. In the sequence (5.1), the element $[1]$ is not contained in the kernel $\mathrm{Aut}(\mathrm{D}^\mathrm{b}(X),\sigma)$. Hence $\mathrm{Aut}(\mathrm{D}^\mathrm{b}(X),\sigma)$, as a subgroup of either $0\times\mathbb{Z}_2[1]$ or $\mathbb{Z}_2\times\mathbb{Z}_2[1]$, can only be trivial or $\mathbb{Z}_2$. □

For K3 surfaces of Picard number one, Lemma 5.2 shows that every finite subgroup of autoequivalences, if nontrivial, is isomorphic to $\mathbb{Z}_2$. Therefore, such subgroups one-to-one correspond to involutions. Thus the classifications of conjugacy classes of finite subgroups and conjugacy classes of involutions are the same. In the following, we will give a classification of the latter starting with some general properties about involutions on a triangulated category.

Lemma 5.3. *Let $\mathscr{D}$ be a triangulated category and $\phi\in\mathrm{Aut}(\mathscr{D})/\mathbb{Z}[2]$ be an involution not represented by a shift functor. Then either ϕ or $\phi[1]$ admits a representative $\Phi\in\mathrm{Aut}(\mathscr{D})$ which is an involution. Moreover, Φ is the only member which has finite order in the set of autoequivalences representing ϕ or $\phi[1]$.*

Proof. Let $\Psi\in\mathrm{Aut}(\mathscr{D})$ be any representative of ϕ. Because ϕ is involutive, $\Psi^2=[2m]$ for some integer m. Define $\Phi:=\Psi[-m]$. Then $\Phi^2=\Psi^2[-2m]=\mathrm{id}$. If $\Phi=\mathrm{id}$, then $\Psi=[m]$, thus ϕ is represented by a shift functor, contradiction. Hence Φ is an involution.

By definition, Φ represents either ϕ or $\phi[1]$ depending on whether m is even or odd. If Φ represents ϕ, then any other representative of ϕ is different from Φ by an even shift, and any

representative of $\phi[1]$ is different from Φ by an odd shift. A similar situation occurs when Φ represents $\phi[1]$. In both cases, Φ is the only element which is of finite order among all the representatives of ϕ and $\phi[1]$. □

By mapping an involution $\phi \in (\mathrm{Aut}(\mathscr{D})/\mathbb{Z}[2]) \setminus \{[1]\}$ to the autoequivalence Φ determined by Lemma 5.3, we obtain a map

$$\{\text{involutions in } \mathrm{Aut}(\mathscr{D})/\mathbb{Z}[2]\} \setminus \{[1]\} \longrightarrow \{\text{involutions in } \mathrm{Aut}(\mathscr{D})\}. \tag{5.3}$$

Notice that this map sends ϕ and $\phi[1]$ to the same involution.

Lemma 5.4. *Map (5.3) is two-to-one and preserves conjugacy classes.*

Proof. For every involution $\Phi \in \mathrm{Aut}(\mathscr{D})$, its image $\overline{\Phi} \in \mathrm{Aut}(\mathscr{D})/\mathbb{Z}[2]$ is an involution not equal to $[1]$. Because Φ is a finite order representative of $\overline{\Phi}$, the element $\overline{\Phi}$ is mapped back to Φ by (5.3). This shows that mapping Φ to $\overline{\Phi}$ defines a section of the map. In particular, the map is surjective.

To prove that the map is two-to-one, pick involutions $\phi, \phi' \in (\mathrm{Aut}(\mathscr{D})/\mathbb{Z}[2]) \setminus \{[1]\}$ and assume that they are mapped to the same autoequivalence $\Phi \in \mathrm{Aut}(\mathscr{D})$ under (5.3). In this setting, Φ represents either ϕ or $\phi[1]$, and either ϕ' or $\phi'[1]$. One can deduce that $\phi' = \phi$ or $\phi' = \phi[1]$ via a case-by-case analysis. This fact, together with the surjectivity proved above, asserts that the map is two-to-one.

To prove that the map preserves conjugacy classes, let $\phi, \phi' \in (\mathrm{Aut}(\mathscr{D})/\mathbb{Z}[2]) \setminus \{[1]\}$ be involutions and $\Phi, \Phi' \in \mathrm{Aut}(\mathscr{D})$ respectively be their images under the map. Assume that there exists $\theta \in \mathrm{Aut}(\mathscr{D})/\mathbb{Z}[2]$ such that $\phi' = \theta\phi\theta^{-1}$. Then, if $\Theta \in \mathrm{Aut}(\mathscr{D})$ represents θ, we have $\Phi' = \Theta\Phi\Theta^{-1}[m]$ for some even integer m. Because Φ and Φ' are involutions, taking squares on both sides of this relation gives $\mathrm{id} = [2m]$, which implies that $m = 0$. Hence we have $\Phi' = \Theta\Phi\Theta^{-1}$, so Φ and Φ' are conjugate to each other. □

Now consider a K3 surface X of odd Picard number. In this case, the only automorphisms of the transcendental lattice are $\pm\mathrm{id}$ [Huy16a, Corollary 3.3.5]. Therefore, every autoequivalence is either symplectic or anti-symplectic, and composing with [1] changes one type to the other. For every $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$, let us denote its image in $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ as $\overline{\Phi}$. Consider the map

$$\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow \mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2] : \Phi \longmapsto \begin{cases} \overline{\Phi} & \text{if } \Phi \text{ is symplectic} \\ \overline{\Phi}[1] & \text{if } \Phi \text{ is anti-symplectic} \end{cases}.$$

Under this map, an involution is mapped to an involution, and conjugate elements are mapped to conjugate elements. Therefore, it induces a map

$$\{\text{involutions in } \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))\} \longrightarrow \{\text{involutions in } \mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]\} \tag{5.4}$$

preserving conjugacy classes. Notice that the codomain does not contain $[1]$, so it appears as a subset of the domain of map (5.3). Restricting (5.3) to this subset gives

$$\{\text{involutions in } \mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]\} \longrightarrow \{\text{involutions in } \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))\} \tag{5.5}$$

which also preserves conjugacy classes.

Lemma 5.5. *Maps* (5.4) *and* (5.5) *are inverse to each other, and they induce bijections between conjugacy classes.*

Proof. The image of an involution $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ under (5.4) is either $\overline{\Phi}$ or $\overline{\Phi}[1]$, which are both mapped to Φ by (5.5). This shows that (5.5) $\circ$ (5.4) $=$ id.

Next, pick an involution $\phi \in \mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ and let $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ be its image under (5.5). Then Φ represents either ϕ or $\phi[1]$.

- If Φ represents ϕ, then it is symplectic, thus (5.4) maps it to $\overline{\Phi} = \phi$.
- If Φ represents $\phi[1]$, then it is anti-symplectic, thus (5.4) maps it to $\overline{\Phi}[1] = \phi$.

This shows that (5.4) $\circ$ (5.5) $=$ id. Thus the two maps are inverse to each other.

Recall that both maps preserve conjugacy classes. Hence they induce maps between conjugacy classes that are inverse to each other. □

Theorem 5.6. *Let X be a K3 surface of Picard number one and degree $2n$. If G is a maximal finite subgroup of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$, then there exists $\sigma \in \mathrm{Stab}^{\dagger}(X)$ such that*

$$G = \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X), \sigma) \cong \begin{cases} \mathbb{Z}_2 & \text{if} \quad n = 1, 2 \\ 0 & \text{if} \quad n = 3, 4 \\ 0,\ \mathbb{Z}_2 & \text{if} \quad n \geq 5. \end{cases}$$

- *When $n = 1, 2$, there exists one and only one such subgroup up to conjugation.*
- *When $n \geq 5$, there exist $\frac{\nu_2}{2}$ many such subgroups isomorphic to $\mathbb{Z}_2$ up to conjugation.*

Proof. The identification $G = \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X), \sigma)$ is given by Corollary 4.14. By Lemma 5.2, the group $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X), \sigma)$ is either trivial or isomorphic to $\mathbb{Z}_2$, which turns the problem into the classification of involutions in $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$. Lemma 5.5 further turns it into the classification of involutions in $\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$. The list and the numbers of conjugacy classes then follows from Lemma 4.8. □

The unique conjugacy classes of involutions when $n = 1, 2$ are induced, respectively, by the covering involution of $X \to \mathbb{P}^2$ and $\Theta^2[-1]$ where $\Theta = (- \otimes \mathcal{O}_X(1)) \circ T_{\mathcal{O}_X}$ (see Example 4.9). Notice that both involutions are anti-symplectic. We will show that, in fact, all involutive autoequivalences are anti-symplectic for all n.

Let us assume that $n \geq 2$ and consider the isometry group

$$\widehat{\mathrm{O}}(\mathcal{N}(X)) := \{\phi \in \mathrm{O}(\mathcal{N}(X)) \mid \phi \text{ acts on } \mathcal{N}(X)^*/\mathcal{N}(X) \text{ as } \pm \mathrm{id}\}.$$

This group admits two natural homomorphisms to $\mathbb{Z}_2 \cong \{\pm 1\}$ given by

$$\det \colon \widehat{\mathrm{O}}(\mathcal{N}(X)) \longrightarrow \{\pm 1\} \qquad \text{and} \qquad \mathrm{disc} \colon \widehat{\mathrm{O}}(\mathcal{N}(X)) \longrightarrow \{\pm 1\}$$

where det is the determinant function and disc takes the sign of the action of an isometry on the discriminant group. Note that elements in (resp. outside) the kernel of disc become symplectic (resp. anti-symplectic) upon extended to $\widetilde{H}(X, \mathbb{Z})$. We have

$$(\det \cdot \mathrm{disc})(\mathrm{id}) = (\det \cdot \mathrm{disc})(-\mathrm{id}) = 1.$$

Thus the multiplication $\det \cdot \operatorname{disc}$ descends to the quotient $\widehat{\mathrm{O}}(\mathcal{N}(X))/\{\pm\mathrm{id}\}$ as

$$\overline{\det \cdot \operatorname{disc}} \colon \widehat{\mathrm{O}}(\mathcal{N}(X))/\{\pm\mathrm{id}\} \longrightarrow \{\pm 1\}.$$

The isomorphism $\mathcal{H} \cong \mathcal{Q}^+(X)$ in (4.1) identifies $\Gamma_0^+(n)$ as a subgroup of $\widehat{\mathrm{O}}(\mathcal{N}(X))/\{\pm\mathrm{id}\}$, so we can consider the restriction

$$\mathbf{d} := \overline{\det \cdot \operatorname{disc}}\,|_{\Gamma_0^+(n)} \colon \Gamma_0^+(n) \longrightarrow \{\pm 1\}.$$

Lemma 5.7. *For every $n \geq 2$, we have $\Gamma_0(n) = \ker(\mathbf{d})$.*

Proof. Pick any $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0^+(n)$. Then the actions on $\mathcal{N}(X)$ which induce g are represented by $\pm M_g$ where

$$M_g = \begin{pmatrix} \delta^2 & 2\gamma\delta & \frac{1}{n}\gamma^2 \\ \beta\delta & \alpha\delta + \beta\gamma & \frac{1}{n}\alpha\gamma \\ n\beta^2 & 2n\alpha\beta & \alpha^2 \end{pmatrix}.$$

A direct computation gives $\det(M_g) = (\alpha\delta - \beta\gamma)^3 = 1$. To prove the lemma, it suffices to prove that $g \in \Gamma_0(n)$ if and only if $\operatorname{disc}(M_g) = 1$.

The discriminant group $\mathcal{N}(X)^*/\mathcal{N}(X)$ is generated by $(0, \frac{1}{2n}, 0)$. One can verify that M_g acts on this generator by multiplying

$$\alpha\delta + \beta\gamma = 2\alpha\delta - 1 = 2\beta\gamma + 1.$$

- If $g \in \Gamma_0(n)$, then $\gamma \equiv 0 \pmod{n}$, thus $2\beta\gamma + 1 \equiv 1 \pmod{2n}$. In this case, the action of M_g on the discriminant group is the identity, so $\operatorname{disc}(M_g) = 1$.
- If $g \notin \Gamma_0(n)$, then we can write

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{\sqrt{n}} \\ \sqrt{n} & 0 \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(n).$$

  From here we get $\alpha\delta = -bc \equiv 0 \pmod{n}$, thus $2\alpha\delta - 1 \equiv -1 \pmod{2n}$. In this case, the action of M_g on the discriminant group is $-\mathrm{id}$, so $\operatorname{disc}(M_g) = -1$.

This shows that $g \in \Gamma_0(n)$ if and only if $\operatorname{disc}(M_g) = 1$, which completes the proof. □

Corollary 5.8. *On a K3 surface X of Picard number one, every involutive autoequivalence is anti-symplectic.*

Proof. Suppose that X has degree $2n$. If $n = 1$, then every involutive autoequivalence is conjugate to the one induced by the covering involution $X \to \mathbb{P}^2$, which is anti-symplectic, so the statement holds in this case.

Assume that $n \geq 2$ and let $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ be an involution. By Theorem 4.13, there exists $\sigma \in \mathrm{Stab}^\dagger(X)$ fixed by Φ, so that the induced isometry $\phi \in \widehat{\mathrm{O}}(\mathcal{N}(X))$ is the reflection across the positive plane determined by σ. This implies $\det(\phi) = -1$. Notice that ϕ cannot

be the reflection along a (-2)-vector since, otherwise, Φ would be the composition of a spherical twist with some element from the Torelli group $\mathcal{I}(\mathrm{D}^{\mathrm{b}}(X))$ and thus would be of infinite order. Hence ϕ corresponds to an involution in $\Gamma_0(n)$ by Lemma 4.2. The fact that $\det(\phi) = -1$ and Lemma 5.7 imply that $\mathrm{disc}(\phi) = -1$. This proves that ϕ, and thus Φ, is anti-symplectic. □

Proof of Theorem 1.1. Merging Corollary 5.8 into Theorem 5.6 gives the statement. □

5.2 Distribution of Gepner type stability conditions Following [BB17], we say a stability condition $\sigma \in \mathrm{Stab}^{\dagger}(X)$ is *reduced* if its central charge Z, regarded as an element of $\mathcal{N}(X) \otimes \mathbb{C}$, satisfies $(Z, Z) = 0$. The set of reduced stability conditions forms a submanifold

$$\mathrm{Stab}^{\dagger}_{\mathrm{red}}(X) \subseteq \mathrm{Stab}^{\dagger}(X)$$

which is invariant under the free $\mathbb{C}$-action. Moreover, we have [BB17, Lemma 2.1]

$$\mathrm{Stab}^{\dagger}_{\mathrm{red}}(X)/\mathbb{C} \;\cong\; \mathrm{Stab}^{\dagger}(X)/\widetilde{\mathrm{GL}}^{+}(2,\mathbb{R}).$$

Lemma 5.9. *Let X be an algebraic K3 surface and pick $\sigma \in \mathrm{Stab}^{\dagger}(X)$. Suppose that there exists $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ which satisfies $\Phi(\sigma) = \sigma \cdot \lambda$ for some $\lambda \in \mathbb{C}\backslash\mathbb{Z}$. Then σ is reduced.*

Proof. Let Z_σ and $Z_{\sigma\cdot\lambda}$ denote the central charges of σ and $\sigma \cdot \lambda$ respectively, so that we have $Z_{\sigma\cdot\lambda} = Z_\sigma \cdot e^{-i\pi\lambda}$. Since autoequivalences preserve pairings between central charges, the condition $\Phi(\sigma) = \sigma \cdot \lambda$ implies that

$$(Z_\sigma, Z_\sigma) = (Z_{\sigma\cdot\lambda}, Z_{\sigma\cdot\lambda}) = (Z_\sigma, Z_\sigma) \cdot e^{-2i\pi\lambda}.$$

The assumption $\lambda \notin \mathbb{Z}$ implies $e^{-2i\pi\lambda} \neq 1$. Hence $(Z_\sigma, Z_\sigma) = 0$, thus σ is reduced. □

Lemma 5.10. *Let X be a K3 surface of Picard number one and $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ be a finite order element. Then Φ fixes a point on $\mathrm{Stab}^{\dagger}_{\mathrm{red}}(X)$.*

Proof. By Theorem 4.13, there exists $\sigma \in \mathrm{Stab}^{\dagger}(X)$ such that $\Phi(\sigma) = \sigma \cdot \lambda$ for some $\lambda \in \mathbb{C}$.

- If $\lambda \notin \mathbb{Z}$, then Lemma 5.9 implies that $\sigma \in \mathrm{Stab}^{\dagger}_{\mathrm{red}}(X)$.
- If $\lambda \in \mathbb{Z}$, then it lives in the center of $\widetilde{\mathrm{GL}}^{+}(2,\mathbb{R})$. Hence $\Phi(\sigma \cdot g) = (\sigma \cdot g) \cdot \lambda$ for every $g \in \widetilde{\mathrm{GL}}^{+}(2,\mathbb{R})$, and we can choose g so that $\sigma \cdot g \in \mathrm{Stab}^{\dagger}_{\mathrm{red}}(X)$.

In both cases, Φ fixes a point on $\mathrm{Stab}^{\dagger}_{\mathrm{red}}(X)/\mathbb{C}$. □

Lemma 5.11. *Let X be a K3 surface of Picard number one and $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ be a finite order element which induces a nontrivial action on $\mathcal{Q}^{+}_0(X)$. Then Φ fixes one and only one point on $\mathrm{Stab}^{\dagger}_{\mathrm{red}}(X)$.*

Proof. The existence of a fixed point is guaranteed by Lemma 5.10. Let us prove that there is only one fixed point. Suppose that $\sigma, \sigma' \in \mathrm{Stab}^\dagger_{\mathrm{red}}(X)$ and $\lambda, \lambda' \in \mathbb{C}$ satisfy

$$\Phi(\sigma) = \sigma \cdot \lambda \qquad \text{and} \qquad \Phi(\sigma') = \sigma' \cdot \lambda'.$$

By hypothesis, the action of Φ on $\mathcal{Q}_0^+(X)$ has a unique fixed point. Thus the orbits $\sigma \cdot \mathbb{C}$ and $\sigma' \cdot \mathbb{C}$ are mapped to this fixed point under the covering map

$$\mathrm{Stab}^\dagger_{\mathrm{red}}(X)/\mathbb{C} \cong \mathrm{Stab}^\dagger(X)/\widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \longrightarrow \mathcal{Q}_0^+(X).$$

Recall that $\mathcal{I}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ is the group of deck transformations of this covering and its action is free. In particular, there exist $\Psi \in \mathcal{I}(\mathrm{D}^{\mathrm{b}}(X))$ and $\lambda'' \in \mathbb{C}$ such that $\sigma' = \Psi(\sigma) \cdot \lambda''$. It follows that

$$\begin{aligned}\Psi^{-1}\Phi\Psi\Phi^{-1}(\sigma) = \Psi^{-1}\Phi\Psi(\sigma) \cdot (-\lambda) &= \Psi^{-1}\Phi(\sigma') \cdot (-\lambda'' - \lambda) \\ &= \Psi^{-1}(\sigma') \cdot (\lambda' - \lambda'' - \lambda) = \sigma \cdot (\lambda' - \lambda).\end{aligned}$$

This shows that $\Psi^{-1}\Phi\Psi\Phi^{-1} \in \mathcal{I}(\mathrm{D}^{\mathrm{b}}(X))$ is of Gepner type with respect to σ, which implies that it acts trivially as a covering transformation, so we have $\Psi^{-1}\Phi\Psi\Phi^{-1} \in \mathbb{Z}[2]$. Let us write $\Psi = T_{S_1}^{2k_1} \cdots T_{S_\ell}^{2k_\ell}[2m]$. Then

$$\Psi^{-1}\Phi\Psi\Phi^{-1} = T_{S_\ell}^{-2k_\ell} \cdots T_{S_1}^{-2k_1} T_{\Phi(S_1)}^{2k_1} \cdots T_{\Phi(S_\ell)}^{2k_\ell} \in \mathbb{Z}[2]. \tag{5.6}$$

The unique fixed point on $\mathcal{Q}_0^+(X)$ under the action of Φ is not given by a (-2)-vector. Hence Φ does not preserve any (-2)-vector, so it does not fix any spherical object. In order for (5.6) to hold, we must have $\ell = 0$, which shows that $\Psi \in \mathbb{Z}[2]$. As a result, σ and σ' represents the same point on $\mathrm{Stab}^\dagger_{\mathrm{red}}(X)/\mathbb{C}$. □

Theorem 5.12. *Let X be a K3 surface of Picard number one and degree $2n$. Then every nontrivial maximal finite subgroup of $\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ fixes one and only one point on $\mathrm{Stab}^\dagger_{\mathrm{red}}(X)/\mathbb{C}$. This property defines a one-to-one correspondence between*

(a) *the set of maximal finite subgroups of $\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ not equal to $\mathbb{Z}_2[1]$, and*

(b) *the set of points on $\mathrm{Stab}^\dagger_{\mathrm{red}}(X)/\mathbb{C}$ over elliptic points of $\Gamma_0^+(n)$ on $\mathcal{Q}_0^+(X)$ under the covering map*

$$\mathrm{Stab}^\dagger_{\mathrm{red}}(X)/\mathbb{C} \longrightarrow \mathcal{Q}_0^+(X).$$

The same statement holds with Aut_s replaced by Aut if we require the maximal finite subgroups to be not equal to $\mathbb{Z}_2[1]$.

Proof. By Lemma 4.8, every nontrivial maximal finite subgroup of $\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ is generated by an element which acts nontrivially on $\mathcal{Q}_0^+(X)$. Therefore, it fixes a unique point on $\mathrm{Stab}^\dagger_{\mathrm{red}}(X)/\mathbb{C}$ by Lemma 5.11. This induces a map from (a) to (b). Let us prove that this map is bijective:

- Injectivity: Let G and G' be nontrivial maximal finite subgroups of $\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ which correspond to the same $\sigma \cdot \mathbb{C} \in \mathrm{Stab}^\dagger_{\mathrm{red}}(X)/\mathbb{C}$. Then both of them appear as maximal finite subgroups in the group $\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X), \sigma \cdot \mathbb{C})/\mathbb{Z}[2]$, which is also finite by Proposition 3.4. This implies that $G = G'$.

- Surjectivity: Pick any $\overline{\sigma} \in \mathrm{Stab}^\dagger_{\mathrm{red}}(X)/\mathbb{C}$ over an elliptic point $p \in \mathcal{Q}^+_0(X)$. Via the isomorphism
$$\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2] \;\cong\; \pi_1^{\mathrm{orb}}(\Gamma_0^+(n) \setminus\!\setminus \mathcal{Q}^+_0(X)),$$
the stabilizer of p in the Fricke group $\Gamma_0^+(n)$ corresponds to a maximal finite cyclic subgroup $G \subseteq \mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ acting nontrivially on $\mathcal{Q}^+_0(X)$. Lemma 5.11 implies that G fixes a point $\overline{\sigma}' \in \mathrm{Stab}^\dagger_{\mathrm{red}}(X)/\mathbb{C}$. Both $\overline{\sigma}$ and $\overline{\sigma}'$ lie over p, so there exists $\Psi \in \mathcal{I}(\mathrm{D}^{\mathrm{b}}(X))$ such that $\overline{\sigma} = \Psi(\overline{\sigma}')$. Then $\Psi G \Psi^{-1}$ gives a maximal finite subgroup fixing $\overline{\sigma}$.

This establishes a one-to-one correspondence between (a) and (b).

Let us prove the statement with Aut_s replaced by Aut. By Lemma 4.12, every maximal finite subgroup $G \subseteq \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ has the form $G \cong G_s \times \mathbb{Z}_2[1]$ where G_s is a maximal finite subgroup of $\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$. The statement then follows from the symplectic version since $\mathrm{Stab}^\dagger_{\mathrm{red}}(X)/\mathbb{C}$ is pointwisely fixed by [1]. □

Theorem 5.13. *Suppose that $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ fixes a unique $\overline{\sigma} \in \mathrm{Stab}^\dagger_{\mathrm{red}}(X)/\mathbb{C}$. Let r be the order of Φ modulo even shifts and $\sigma \in \mathrm{Stab}^\dagger(X)$ be a representative of $\overline{\sigma}$.*

- *If $r = 2$, then every point on $\sigma \cdot \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ is of Gepner type with respect to Φ.*
- *If $r \geq 3$, then only the points on $\sigma \cdot \mathbb{C}$ are of Gepner type with respect to Φ.*

Proof. By Corollary 3.5, there exists an integer k such that
$$\Phi(\sigma) = \sigma \cdot \frac{2k}{r}. \tag{5.7}$$
Composing Φ with a shift functor does not change the Gepner type stability conditions corresponding to it, so we can assume $0 \leq 2k < r$. If $r = 2$, then $k = 0$, thus (5.7) becomes $\Phi(\sigma) = \sigma$. Hence $\Phi(\sigma \cdot g) = \sigma \cdot g$ for all $g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$. This proves the first statement.

Assume that $r \geq 3$. If $k = 0$, then Φ fixes σ, which implies that Φ is an involution by Lemma 5.2 and our hypothesis, but this means $r = 2$, contradiction. Hence $k > 0$. Now, as $0 < \frac{2k}{r} < 1$, the centralizer of $\frac{2k}{r}$ in $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ is equal to $\mathbb{C}$ by Lemma 2.2. This completes the proof. □

5.3 Polychotomy of autoequivalences and categorical entropy Let X be a K3 surface of Picard number one and degree $2n$. Recall from Section 4.1 that the actions of autoequivalences on the hyperbolic plane $\mathcal{Q}^+(X) \cong \mathcal{H}$ induces a surjective homomorphism
$$F\colon\ \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \twoheadrightarrow \Gamma_0^+(n).$$
Let us denote its kernel as
$$\widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X)) := \ker(F) = \begin{cases} \langle \mathcal{I}(\mathrm{D}^{\mathrm{b}}(X)), [1] \rangle & \text{if} \quad n \geq 2 \\ \langle \mathcal{I}(\mathrm{D}^{\mathrm{b}}(X)), [1], \iota \rangle & \text{if} \quad n = 1 \end{cases}$$
where, in the case that $n = 1$, the autoequivalence ι is the one induced by the covering involution of $X \to \mathbb{P}^2$.

Recall that the action on $\mathcal{N}(X)$ induced by $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \Gamma_0^+(n)$ is represented by the matrices $\pm M$ where

$$M = \begin{pmatrix} \delta^2 & 2\gamma\delta & \frac{1}{n}\gamma^2 \\ \beta\delta & \alpha\delta+\beta\gamma & \frac{1}{n}\alpha\gamma \\ n\beta^2 & 2n\alpha\beta & \alpha^2 \end{pmatrix}$$

Lemma 5.14. *Suppose that $M \neq \mathrm{id}$. Then:*

- *The eigenvalues of M are*

$$\lambda = 1 \qquad \text{and} \qquad \mu^{\pm} = \frac{-2+(\alpha+\delta)^2 \pm i(\alpha+\delta)\sqrt{4-(\alpha+\delta)^2}}{2}.$$

- *The 1-eigenspace of M is one-dimensional, and is generated by the vector*

$$v = \begin{pmatrix} 2\gamma \\ \alpha-\delta \\ -2n\beta \end{pmatrix}.$$

  In particular, when $|\alpha+\delta| = 2$ where the only eigenvalue of M is 1, it has only one Jordan block which is of size three.

- *The self-pairing of v, when considered as a vector in $\mathcal{N}(X)\otimes\mathbb{R}$, is*

$$v^2 = 2n\left((\alpha+\delta)^2 - 4\right).$$

Proof. The eigenvalues of M can be computed directly, and it is easy to verify that $Mv = v$. When $|\alpha+\delta| = 2$, assume without loss of generality that $\beta \neq 0$. Then we have

$$\begin{pmatrix} \delta^2 & 2\gamma\delta & \frac{1}{n}\gamma^2 \\ \beta\delta & \alpha\delta+\beta\gamma & \frac{1}{n}\alpha\gamma \\ n\beta^2 & 2n\alpha\beta & \alpha^2. \end{pmatrix} = \begin{pmatrix} \frac{(1-\alpha)^2}{\beta^2 n} & \frac{1-\alpha}{\beta^2 n} & \frac{\alpha}{2\beta^2 n} \\ \frac{1-\alpha}{\beta n} & \frac{1}{2\beta n} & -\frac{1}{4\beta n} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{(1-\alpha)^2}{\beta^2 n} & \frac{1-\alpha}{\beta^2 n} & \frac{\alpha}{2\beta^2 n} \\ \frac{1-\alpha}{\beta n} & \frac{1}{2\beta n} & -\frac{1}{4\beta n} \\ 1 & 0 & 0 \end{pmatrix}^{-1} \tag{5.8}$$

Finally, $v^2 = 2n\left((\alpha+\delta)^2-4\right)$ follows from the condition that $\alpha\delta - \beta\gamma = 1$. □

Proposition 5.15. *Let X be a K3 surface of Picard number one and Φ be an autoequivalence not contained in $\widetilde{\mathcal{I}}(\mathrm{D^b}(X))$. Denote by $[\Phi]$ the isometry on $\mathcal{N}(X)$ induced by Φ. Then exactly one of the following holds:*

(a) *There exists $\Psi \in \widetilde{\mathcal{I}}(\mathrm{D^b}(X))$ such that $(\Phi\Psi)^m = [k]$ for some $m > 0$ and $k \in \mathbb{Z}$.*

(b) *There exists $\Psi \in \widetilde{\mathcal{I}}(\mathrm{D^b}(X))$ and a spherical object $S \in \mathrm{D^b}(X)$ such that $\Phi\Psi = T_S$.*

(c) *There exists a nonzero vector $w \in \mathcal{N}(X)$ such that $w^2 = 0$ and $[\Phi]w = w$.*

(d) *$\rho([\Phi]) > 1$, where ρ is the spectral radius of $[\Phi]$.*

We say Φ is of finite order type, (-2)-reducible type, 0-reducible type, or pseudo-Anosov type, if it satisfies (a), (b), (c), or (d), respectively.

Proof. Let us classify Φ according to whether $F(\Phi) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0^+(n)$ is elliptic, parabolic, or hyperbolic. First assume that $F(\Phi)$ is elliptic. Then its action on $\mathcal{Q}^+(X)$ has a fixed point:

- The fixed point belongs to $\mathcal{Q}_0^+(X)$ if and only if there exists $\Psi \in \widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$ such that $\Phi\Psi$ has a fixed point on $\mathrm{Stab}^\dagger_{\mathrm{red}}(X)/\mathbb{C}$ by [Bri08, Theorem 1.1]. This is equivalent to condition (a) by Theorem 5.12.
- The fixed point point belongs to $\mathcal{Q}^+(X) \backslash \mathcal{Q}_0^+(X)$ if and only if $F(\Phi)$ is an involution with respect to a (-2)-point, which is equivalent to (b) by Proposition 4.3.

It remains to consider the cases where $F(\Phi)$ is parabolic or hyperbolic.

- $F(\Phi)$ is hyperbolic, i.e. $|\alpha + \delta| > 2$, is equivalent to (d) by Lemma 5.14.
- $F(\Phi)$ is parabolic, i.e. $|\alpha + \delta| = 2$, is equivalent to the condition that there exists a nonzero $v \in \mathcal{N}(X) \otimes \mathbb{R}$ so that $[\Phi]v = v$ and $v^2 = 0$ by Lemma 5.14. Moreover, the vector
$$v = \begin{pmatrix} 2\gamma \\ \alpha - \delta \\ -2n\beta \end{pmatrix}$$
is integral in this case.

This completes the proof. □

Remark 5.16. Proposition 5.15 resembles the *Nielsen–Thurston classification* of mapping classes on Riemann surfaces, where mapping classes are classified into three types: finite order, reducible, or pseudo-Anosov. A similar polychotomy in the case of elliptic curves was established in [KKO22, Theorem 5.20]. In our setting, the reducible autoequivalences are further classified into (-2)-*reducible* and 0-*reducible* types which corresponds respectively to the (-2)-*points* and the *cusps* of the modular curve $\Gamma_0^+(n) \setminus\!\setminus \mathcal{Q}^+(X)$.

Unlike (-2)-reducible autoequivalences, which are spherical twists (composing with an element of $\widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$), the 0-reducible autoequivalences are less known. For each nonzero vector $w \in \mathcal{N}(X)$ with $w^2 = 0$, let us define

$$\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X), w) := \left\{ \Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)) \mid [\Phi]w = w \right\}.$$

Proposition 5.17. *Let X be a K3 surface of Picard number one and $w \in \mathcal{N}(X)$ be a nonzero vector with $w^2 = 0$. If X has degree at least 4, then there is a short exact sequence*

$$0 \longrightarrow \mathcal{I}(\mathrm{D}^{\mathrm{b}}(X)) \longrightarrow \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X), w) \xrightarrow{F} \mathbb{Z} \longrightarrow 0.$$

If X has degree 2, then the kernel should be replaced by $\langle \mathcal{I}(\mathrm{D}^{\mathrm{b}}(X)), \iota \rangle$.

Proof. The kernel of the map $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X), w) \longrightarrow \Gamma_0^+(n)$ contains $\mathcal{I}(\mathrm{D}^{\mathrm{b}}(X))$ and is contained in $\widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$. In the case of degree at least 4, it coincides with $\mathcal{I}(\mathrm{D}^{\mathrm{b}}(X))$ because $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X), w)$ does not contain [1]. In the case of degree 2, the involution ι belongs to $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X), w)$ since it acts trivially on $\mathcal{N}(X)$.

We claim that the image of the map $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X), w) \longrightarrow \Gamma_0^+(n)$ is an infinite cyclic group that fixes a cusp of $\Gamma_0^+(n) \setminus\!\setminus \mathcal{Q}_0^+(X)$. Write

$$w = (r, d, s) \in \mathcal{N}(X)\backslash\{0\} \qquad \text{where} \qquad w^2 = 2(nd^2 - rs) = 0.$$

Assume $s \neq 0$ without loss of generality. Then one can easily see that any non-identity isometry $[\Phi]$ on $\mathcal{N}(X)$ that fixes w must have $\beta \neq 0$. It follows from the decomposition (5.8) that such $[\Phi]$ satisfies the condition:

$$\begin{pmatrix} \frac{(1-\alpha)^2}{\beta^2 n} \\ \frac{1-\alpha}{\beta n} \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} r \\ d \\ s \end{pmatrix} \text{ are parallel, or equivalently, } \frac{1-\alpha}{\beta} = \frac{nd}{s}.$$

Hence the image of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X), w) \longrightarrow \Gamma_0^+(n)$ equals

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & 2-\alpha \end{pmatrix} \;\middle|\; \frac{1-\alpha}{\beta} = \frac{nd}{s} \right\},$$

which is precisely the subgroup of $\Gamma_0^+(n)$ fixing the cusp at $\frac{s}{nd}$. □

Example 5.18. By Proposition 5.17, every autoequivalence fixing $w = (0, 0, 1)$ is of the form $\Psi \circ (- \otimes \mathcal{O}(n))$ for some $\Psi \in \mathcal{I}(\mathrm{D}^{\mathrm{b}}(X))$ and $n \in \mathbb{Z}$. Note that $- \otimes \mathcal{O}(1)$ fixes not only w but also skyscraper sheaves, which are semirigid objects representing w.

In light of this example, we propose the following question:

Question 5.19. Let X be a K3 surface of Picard number one and $w \in \mathcal{N}(X)$ be a primitive nonzero vector with $w^2 = 0$. Do there exist a semirigid object $E \in \mathrm{D}^{\mathrm{b}}(X)$ and an autoequivalence Φ not contained in $\widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$ such that $v(E) = w$ and $\Phi(E) = E$?

Another way to understand the polychotomy is via dynamical behaviors of autoequivalences, that is, the behaviour of iterations Φ^n when n tends to infinity. There are various invariants that one can associate to autoequivalences from dynamical perspective. Among them, the *categorical entropy* h_{cat} and *categorical polynomial entropy* h_{poly} introduced in [DHKK14, FFO21] are most closely related to the polychotomy. For instance, it is proved in [FFO21, Theorem 1.4] that, if Φ is an autoequivalence of the bounded derived category of coherent sheaves on an elliptic curve, then

(finite order) $|\mathrm{tr}([\Phi])| < 2$ if and only if $h_{\mathrm{cat}}(\Phi) = h_{\mathrm{poly}}(\Phi) = 0$.

(reducible) $|\mathrm{tr}([\Phi])| = 2$ if and only if $h_{\mathrm{cat}}(\Phi) = 0$ and $h_{\mathrm{poly}}(\Phi) = 1$.

(pseudo-Anosov) $|\mathrm{tr}([\Phi])| > 2$ if and only if $h_{\mathrm{cat}}(\Phi) > 0$.

In the case of K3 surfaces of Picard number one, we propose the following questions concerning the categorical (polynomial) entropy of *reducible* autoequivalences.

Question 5.20. Let X be a K3 surface of Picard number one, $S \in \mathrm{D}^{\mathrm{b}}(X)$ be a spherical object, and $w \in \mathcal{N}(X)$ be a nonzero vector with $w^2 = 0$.

(i) Is it true that $h_{\mathrm{poly}}(T_S\Psi) > 0$ for any $\Psi \in \widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$ satisfying $h_{\mathrm{cat}}(T_S\Psi) = 0$?

(ii) Does there exist $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X), w)$ such that $F(\Phi)$ generates the infinite cyclic group

$$\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X), w)/\ker(F)$$

and satisfies $h_{\mathrm{cat}}(\Phi) = 0$?

Note that $- \otimes \mathcal{O}(1)$ gives an example of (ii) in the case that $w = (0, 0, 1)$.

Positive answers to both questions in Question 5.20 would lead to a categorical trichotomy of autoequivalences in terms of entropy:

Proposition 5.21. *Assuming both questions in Question 5.20 have affirmative answers. Then for every autoequivalence Φ not lying in $\widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$:*

(a) *Φ is of finite order type if and only if there exists $\Psi \in \widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$ such that $h_{\mathrm{cat}}(\Phi\Psi) = h_{\mathrm{poly}}(\Phi\Psi) = 0$.*

(b) *Φ is reducible (either (-2)-reducible or 0-reducible) if and only if*

- *there exists $\Psi \in \widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$ such that $h_{\mathrm{cat}}(\Phi\Psi) = 0$, and*
- *for every $\Psi' \in \widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$ satisfying $h_{\mathrm{cat}}(\Phi\Psi') = 0$, we have $h_{\mathrm{poly}}(\Phi\Psi') > 0$.*

(c) *Φ is pseudo-Anosov if and only if $h_{\mathrm{cat}}(\Phi\Psi) > 0$ for any $\Psi \in \widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$.*

Furthermore, the categorical polynomial entropy can be used to distinguish the (-2)-reducible and the 0-reducible autoequivalences as follows:

(i) *If Φ is (-2)-reducible, then there exists $\Psi \in \widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$ such that $h_{\mathrm{cat}}(\Phi\Psi) = 0$ and $0 < h_{\mathrm{poly}}(\Phi\Psi) \leq 1$.*

(ii) *If Φ is 0-reducible, then any $\Psi \in \widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$ satisfying $h_{\mathrm{cat}}(\Phi\Psi) = 0$ has polynomial entropy $h_{\mathrm{poly}}(\Phi\Psi) \geq 2$.*

Proof. Since both sides of Statements (a)–(c) are mutually exclusive by Proposition 5.15, it suffices to prove their "only if" parts.

- The "only if" part of (a) follows directly from the definition of categorical (polynomial) entropy [DHKK14, FFO21].
- The "only if" part of (c) follows from the Yomdin-type lower bound of categorical entropy $h_{\mathrm{cat}}(\Phi) \geq \log \rho([\Phi])$ ([KST20, Theorem 2.13], [FFO21, Proposition 4.3]).

- If Φ is (-2)-reducible, that is, there exists $\Psi \in \widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$ such that $\Phi\Psi = T_S$ for some spherical object S. Then $h_{\mathrm{cat}}(\Phi\Psi) = h_{\mathrm{cat}}(T_S) = 0$ by [Ouc20, Theorem 1.4], and $h_{\mathrm{poly}}(\Phi\Psi') > 0$ for any $\Psi' \in \widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$ satisfying $h_{\mathrm{cat}}(\Phi\Psi') = 0$ assuming Question 5.20 (i) has an affirmative answer.
- If Φ is 0-reducible, then there exists $\Psi \in \widetilde{\mathcal{I}}(\mathrm{D}^{\mathrm{b}}(X))$ such that $h_{\mathrm{cat}}(\Phi\Psi) = 0$ assuming Question 5.20 (ii) has affirmative answer. Furthermore, any $\Psi \in \mathcal{I}(\mathrm{D}^{\mathrm{b}}(X))$ satisfying $h_{\mathrm{cat}}(\Phi\Psi) = 0$ has polynomial entropy $h_{\mathrm{poly}}(\Phi\Psi) \geq 2$ by [FFO21, Proposition 4.5] and Lemma 5.14.

The last part of the statement follows from the fact that $h_{\mathrm{poly}}(T_S) \leq 1$ for any spherical twist T_S [FFO21, Proposition 6.13]. □

6 Criteria for the existence of associated cubic fourfolds

In this section, we develop a few criteria for the existence of a cubic fourfold associated to a K3 surface of Picard number one. Let us start by recalling necessary background about cubic fourfolds and their relations to K3 surfaces.

Let $Y \subseteq \mathbb{P}^5$ be a smooth cubic hypersurface. For a very general Y, the lattice

$$H^{2,2}(Y,\mathbb{Z}) := H^{2,2}(Y,\mathbb{C}) \cap H^4(Y,\mathbb{Z})$$

is spanned by the square of the hyperplane class h^2. Following Hassett [Has00], we say Y is *special* if there exists a rank 2 primitive sublattice $L \subseteq H^4(Y,\mathbb{Z})$ such that

$$h^2 \in L \subseteq H^{2,2}(Y,\mathbb{Z}).$$

The lattice L is called a *labelling*. Special cubic fourfolds with a labelling of discriminant d form an irreducible divisor $\mathcal{C}_d$ in the moduli space of cubic fourfolds, which is nonempty if and only if [Has00, Theorem 4.3.1]

$$d \geq 8 \quad \text{and} \quad d \equiv 0, 2 \pmod 6. \tag{6.1}$$

For a very general $Y \in \mathcal{C}_d$ with a labelling L, it holds that $L = H^{2,2}(X,\mathbb{Z})$.

A K3 surface X and a cubic fourfold Y are *associated* if there are a polarization f on X and a labelling L on Y such that there exists a Hodge isometry

$$f^{\perp_{H^2(X,\mathbb{Z})}(-1)} \xrightarrow{\sim} L^{\perp_{H^4(Y,\mathbb{Z})}}. \tag{6.2}$$

Notice that, if X has Picard number one, then this gives an isometry between the transcendental parts $T(X)$ and $T(Y)$ in their middle cohomologies up to a twist by -1. A cubic fourfold of discriminant d has an associated K3 surface if and only if [Has00, Theorem 5.1.3]

$$d \text{ is not divisible by } 4, 9, \text{ or any odd prime } p \equiv 2 \pmod 3. \tag{6.3}$$

The condition for a K3 surface and a cubic fourfold to be associated is categorical. More precisely, for every cubic fourfold Y, there exists a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(Y) \cong \langle \mathcal{A}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle$$

where the subcategory $\mathcal{A}_Y$ is called the *K3 category* of Y. Then Y is associated Hodge theoretically with a K3 surface X if and only if $\mathcal{A}_Y \cong \mathrm{D}^{\mathrm{b}}(X)$ [AT14, BLM+21].

Proposition 6.1. *Let X be a K3 surface of Picard number one and degree $2n \geq 4$. Then the following conditions are equivalent:*

(1) *There exists a cubic fourfold $Y \subseteq \mathbb{P}^5$ associated with X.*

(2) *There exists $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ satisfying $\Phi^3 = [2]$.*

(3) *$n \geq 7$ and there exists $\phi \in \mathrm{Aut}^+(\widetilde{H}(X,\mathbb{Z}))$ of order 3.*

(4) *$n \geq 7$ and there exists a cyclic subgroup in $\Gamma_0^+(n)$ of order 3, that is, $\nu_3 \neq 0$.*

Proof. The implication (1) $\Rightarrow$ (2) is a consequence of [Kuz04, Lemma 4.2]. Suppose that there exists $\Phi \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X))$ such that $\Phi^3 = [2]$.

- If Φ is symplectic, then it defines an element $\overline{\Phi} \in \mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ of order 3, from which one can check that $n \geq 7$ using (4.7).
- If Φ is anti-symplectic, then $\overline{\Phi}[1] \in \mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X))/\mathbb{Z}[2]$ has order 6. But there is no such element again by (4.7).

This shows that $n \geq 7$. Note that $\Phi \notin \mathcal{I}(\mathrm{D}^{\mathrm{b}}(X))$ because $\mathcal{I}(\mathrm{D}^{\mathrm{b}}(X))$ is free. Hence Φ descends to $\phi \in \mathrm{Aut}^+(\widetilde{H}(X,\mathbb{Z}))$ of order 3. This proves (2) $\Rightarrow$ (3). Now, assume that $n \geq 7$ and that there exists $\phi \in \mathrm{Aut}^+(\widetilde{H}(X,\mathbb{Z}))$ of order 3. Because the group $\mathrm{Aut}^+(\widetilde{H}(X,\mathbb{Z}))$ is surjective onto $\Gamma_0^+(n)$ with kernel equal to $\{\pm\mathrm{id}\}$, the isometry ϕ generates a cyclic subgroup in $\Gamma_0^+(n)$ of order 3, whence $\nu_3 \neq 0$. This proves (3) $\Rightarrow$ (4).

Finally, let us show that (4) $\Rightarrow$ (1). Suppose that $\nu_3 \neq 0$ and define $d := 2n$. Using the formula in [Shi71, Proposition 1.43], one can verify directly that (6.3) holds. We claim that $d \equiv 0, 2 \pmod 6$, or equivalently, $n \equiv 0, 1 \pmod 3$. Indeed, if n contains a prime factor $p \equiv 2 \pmod 3$, then $\nu_3 = 0$ by [Shi71, Proposition 1.43], contradiction. Hence every prime factor of n is congruent to $0, 1 \pmod 3$, thus $n \equiv 0, 1 \pmod 3$. Together with the condition $n \geq 7$, we conclude that (6.1) holds as well. Conditions (6.3) and (6.1) imply that, for all labelled cubic fourfolds (Y, L) such that $\mathrm{disc}(L) = d$ and $L = H^{2,2}(Y,\mathbb{Z})$, there exists an isomorphism (6.2) at the *lattice* level [Has16, Proposition 21]. The surjectivity of the period map for cubic fourfolds [Laz10, Theorem 1.1] then implies that there exists one such Y with (6.2) preserving Hodge structure. □

Remark 6.2. Proposition 6.1 can be extended to K3 surfaces of Picard number one and degree 2. Indeed, such K3 surfaces are associated with cubic fourfolds with an ordinary double point in the sense of [Has00, §4.2]. In this case, the order 3 elements in (2), (3), (4) can be taken to be the ones induced by Θ^2 with the Θ in Example 4.9.

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