A GEOMETRIC APPROACH TO THE DENSITY OF RANK-METRIC CODES

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Abstract. We study the asymptotic density of $\mathbb{F}_q$-point-free linear sections of geometrically irreducible projective varieties over finite fields. We then apply these results to rank-metric codes via determinantal varieties. Our approach recovers the known cases in which the density tends to 0 or 1 and determines the limit in the cases where it was previously unknown. To compute these previously unknown limits, we extend the notion of quasireflexivity to higher-dimensional varieties and show that determinantal varieties satisfy this property. This allows us to invoke the Chebotarev density theorem for varieties over finite fields to obtain the desired estimate.

1. Introduction

Let $\mathbb{F}_q$ be the finite field with q elements, where q is a prime power, and regard $\mathbb{F}_q^{mn}$ as the space of $m \times n$ matrices over $\mathbb{F}_q$ equipped with the rank metric

$$d(A, B) := \operatorname{rank}(A - B), \qquad A, B \in \mathbb{F}_q^{mn}.$$

Throughout the paper, we assume without loss of generality that $m \geq n$. A *linear rank-metric code* is a linear subspace $V \subseteq \mathbb{F}_q^{mn}$. One of the most important invariants of a nonzero code is its *minimum (rank) distance*, defined by

$$\begin{aligned} d_R(V) &:= \min \{d(A, B) \mid A, B \in V,\ A \neq B\} \\ &= \min \{\operatorname{rank}(M) \mid M \in V \setminus \{0\}\}. \end{aligned}$$

The minimum distance determines the error-correcting capabilities of the code.

Rank-metric codes have become a central object of study in modern coding theory. They arise naturally in contexts where errors are measured by rank, with important applications to random network coding and secure network coding [SKK08, SK11]. They are also closely related to several areas of pure mathematics, including q-polymatroids and finite geometry [GJLR20, CMPZ17]. For a general introduction to the subject, we refer the reader to the surveys [GKR23, BHL$^+$22].

The main problem that we study is motivated by analogous results for linear codes in the Hamming metric. There has been extensive study of the distribution of the minimum distance among all k-dimensional linear codes $C \subseteq \mathbb{F}_q^n$. For a recent notable example, see [HHLT22]. Much of the work in this area has focused on the asymptotic problem where q is fixed, while k and n grow together in such a way that k/n converges. Loidreau studied the analogue of this problem for rank-metric codes in [Loi14].

We are interested in a different kind of limit where q tends to infinity while the other parameters remain fixed. Recall that if $C \subseteq \mathbb{F}_q^n$ is a k-dimensional linear code with minimum Hamming distance d, then the Singleton bound gives $d \leq n - k + 1$. Codes for which equality holds are called *MDS* (maximum distance separable). When $q \to \infty$ with k and n fixed,

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it is not difficult to show that most k-dimensional linear codes $C \subseteq \mathbb{F}_q^n$ are MDS. For a quantitative result in this direction, see [Kai14].

We phrase our main results in terms of the following density. Fix integers $1 \leq k \leq mn$ and $\delta \geq 1$, and consider the density of k-dimensional codes with minimum rank distance at least δ:

$$D_q(m \times n, k, \delta) := \frac{\left|\{\text{codes } V \subseteq \mathbb{F}_q^{mn} \mid \dim V = k,\ d_R(V) \geq \delta\}\right|}{\left|\{\text{codes } V \subseteq \mathbb{F}_q^{mn} \mid \dim V = k\}\right|}.$$

The rank-metric analogue of the Singleton bound asserts that for a k-dimensional linear rank-metric code $V \subseteq \mathbb{F}_q^{mn}$, we have $k \leq m(n - d_R(V) + 1)$ [GKR23, Theorem 3.1]. Codes that reach equality in this bound are called *MRD* (maximum rank distance). They are the rank-metric analogues of MDS codes.

There has been significant recent interest in density questions for MRD codes. For an overview of this topic, see [GKR23, Section 6]. Gruica and Ravagnani have determined the asymptotic density of MRD codes for all sets of parameters.

Theorem 1.1 ([GR22, Theorem 5.9])**.** *Assume that $1 \leq \delta \leq n$. Then*

$$\lim_{q \to \infty} D_q(m \times n,\, m(n - \delta + 1),\, \delta) = \begin{cases} 1 & \text{if } \delta = 1, \\ \sum_{i=0}^{m} \frac{(-1)^i}{i!} & \text{if } n = \delta = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The $n = \delta = 2$ case is due to Antrobus and Gluesing-Luerssen [AGL19, Corollary VII.5].

Gruica and Ravagnani apply their techniques to obtain upper and lower bounds on the density of rank-metric codes with more general minimum distance and dimension. This allows them to determine the behavior of $D_q(m \times n, k, \delta)$ as $q \to \infty$ in most cases.

Theorem 1.2 ([GR22, Theorem 5.12])**.** *Assume that $1 \leq k \leq mn$ and $1 \leq \delta \leq n$. Then*

$$\lim_{q \to \infty} D_q(m \times n, k, \delta) = \begin{cases} 0 & \text{if} \quad (\delta - 1)(m + n - \delta + 1) > mn - k + 1, \\ 1 & \text{if} \quad (\delta - 1)(m + n - \delta + 1) < mn - k + 1. \end{cases}$$

In view of this result, we refer to the first situation as the *sparse case* and the second as the *abundant case*. The remaining situation is the *threshold case*, namely when

$$(\delta - 1)(m + n - \delta + 1) = mn - k + 1.$$

Gruica and Ravagnani explain that in the threshold case, [GR22, Proposition 4.4] implies the upper bound

$$\limsup_{q \to \infty} D_q(m \times n, k, \delta) \leq \frac{1}{2}.$$

They further remark that their techniques do not determine the limit in this case and that, in general, rank-metric codes in this setting are neither sparse nor dense as q grows [GR22, Remark 5.13].

1.1. **Main results.** In this paper, we adopt an algebro-geometric approach to these density questions. Our main theorem (Theorem 1.5) resolves the threshold case as a corollary, and also yields concise proofs for the sparse and abundant cases.

Corollary 1.3. *Fix integers $m \geq n$, $2 \leq \delta \leq n$, and $1 \leq k \leq mn$, and set*

$$\Delta := \prod_{i=0}^{n-\delta} \frac{\binom{m+i}{\delta-1}}{\binom{\delta-1+i}{\delta-1}}.$$

Then the asymptotic density of k-dimensional codes in $\mathbb{F}_q^{mn}$ with minimum rank distance at least δ is given by

$$\lim_{q\to\infty} D_q(m \times n, k, \delta) = \begin{cases} 0 & \text{if} \quad (\delta-1)(m+n-\delta+1) > mn-k+1, \\ 1 & \text{if} \quad (\delta-1)(m+n-\delta+1) < mn-k+1, \\ \sum_{i=0}^{\Delta} \frac{(-1)^i}{i!} & \text{if} \quad (\delta-1)(m+n-\delta+1) = mn-k+1. \end{cases}$$

For completeness, when $1 \leq k \leq mn$, we have $D_q(m \times n, k, 1) = 1$, while $D_q(m \times n, k, \delta) = 0$ for every $\delta > n$.

Our main theorem concerns the density of linear subspaces whose intersections with a geometrically irreducible projective variety contain no $\mathbb{F}_q$-rational points. We apply this result to determinantal varieties in order to study rank-metric codes. We now introduce the background necessary to state this result precisely. Let δ be an integer satisfying $2 \leq \delta \leq n$, and set $W := \mathrm{Mat}_{m\times n}(\mathbb{F}_q)$. Inside $\mathbb{P}(W) \cong \mathbb{P}^{mn-1}_{\mathbb{F}_q}$, let Y_δ be the projective determinantal variety cut out by the $\delta \times \delta$ minors. More concretely, we can describe the points of Y_δ as follows. For every field extension $K/\mathbb{F}_q$, its K-points are

$$Y_\delta(K) = \left\{ [M] \in \mathbb{P}^{mn-1}(K) \;\middle|\; \mathrm{rank}_K(M) < \delta \right\}.$$

By definition, a code $V \subseteq \mathbb{F}_q^{mn}$ has $d_R(V) \geq \delta$ if and only if the linear section $\mathbb{P}(V) \cap Y_\delta$ contains no $\mathbb{F}_q$-point. Moreover, [Har92, Proposition 12.2] gives

$$\begin{aligned} \dim Y_\delta &= (mn-1) - (m-\delta+1)(n-\delta+1) \\ &= (\delta-1)(m+n-\delta+1) - 1. \end{aligned}$$

The dimension of a variety can be characterized by its general linear sections [Har92, Definitions 11.2–11.3]. Accordingly, for a general linear subspace $V \subseteq \overline{\mathbb{F}_q}^{mn}$ of dimension k,

$$\mathbb{P}(V) \cap (Y_\delta)_{\overline{\mathbb{F}_q}} \;\text{ is }\; \begin{cases} \text{positive-dimensional} & \text{if} \quad (\delta-1)(m+n-\delta+1) > mn-k+1, \\ \text{empty} & \text{if} \quad (\delta-1)(m+n-\delta+1) < mn-k+1, \\ \text{zero-dimensional} & \text{if} \quad (\delta-1)(m+n-\delta+1) = mn-k+1, \end{cases}$$

which correspond precisely to the sparse, abundant, and threshold cases, respectively.

Determinantal varieties are examples of *geometrically irreducible varieties*, meaning that they remain irreducible in the Zariski topology after extending the base field to the algebraic closure $\overline{\mathbb{F}_q}$; see, for example, [Har92, Proposition 12.2]. They also satisfy *quasireflexivity*, a notion introduced for curves by Entin [Ent21, Section 2]. More precisely, a possibly reducible curve $C \subseteq \mathbb{P}^n$ of degree d is quasireflexive if, for every irreducible component C_i of C, there exists a hyperplane tangent to C_i whose set-theoretic intersection with C consists of $d-1$ geometric points. Thus, the tangency is as simple as possible, while all other intersections are transverse. We extend this notion to higher dimensions, restricting attention to geometrically irreducible varieties.

Definition 1.4. Let K be a field, and let $Y \subseteq \mathbb{P}^N_K$ be a geometrically irreducible projective variety of degree $\Delta \geq 2$. We say that Y is *quasireflexive* if there exists a linear subspace

$L \subseteq \mathbb{P}^N_{\overline{K}}$ of dimension $N - \dim Y$ such that the scheme-theoretic intersection $Z := L \cap Y_{\overline{K}}$ is zero-dimensional and its support consists of exactly $\Delta - 1$ distinct geometric points, all lying in the smooth locus of $Y_{\overline{K}}$.

Since Z is a proper complementary-dimensional linear section, Bézout's theorem gives

$$\sum_{y \in \operatorname{Supp}(Z)} \dim_{\overline{K}} \mathcal{O}_{Z,y} = \Delta.$$

Consequently, after relabeling the points as $\operatorname{Supp}(Z) = \{y_0, y_1, \ldots, y_{\Delta-2}\}$, we have

$$\dim_{\overline{K}} \mathcal{O}_{Z,y_0} = 2 \qquad \text{and} \qquad \dim_{\overline{K}} \mathcal{O}_{Z,y_i} = 1 \quad \text{for } 1 \le i \le \Delta - 2.$$

Thus, exactly one intersection point has multiplicity two, and all the remaining intersections are transverse.

Quasireflexivity enables us to apply the *Chebotarev density theorem*; see Section 3 for further details. To make the asymptotic statements below precise, we work with varieties obtained by base change from a fixed scheme over the integers. Let

$$\mathscr{Y} \subseteq \mathbb{P}^N_{\mathbb{Z}}$$

be a fixed closed projective subscheme defined by finitely many homogeneous polynomials with integer coefficients. For each prime power $q = p^r$, let

$$Y_q := \mathscr{Y} \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Spec}(\mathbb{F}_q) \subseteq \mathbb{P}^N_{\mathbb{F}_q},$$

where the fiber product is taken with respect to the canonical map $\operatorname{Spec}(\mathbb{F}_q) \to \operatorname{Spec}(\mathbb{Z})$. Thus, the varieties Y_q are the finite-field base changes of the single fixed scheme $\mathscr{Y}$.

We assume that, for every prime power q, the scheme Y_q is a geometrically irreducible projective variety, and that its dimension d and degree Δ are independent of q. All limits $q \to \infty$ below are taken over prime powers. When a single prime power q is fixed, we may abbreviate Y_q to Y.

The main result of this paper is as follows.

Theorem 1.5. *Let $\mathscr{Y} \subseteq \mathbb{P}^N_{\mathbb{Z}}$ and its base changes Y_q be as above, so that every Y_q is a geometrically irreducible projective variety of dimension d and degree Δ. Let $0 \le \ell \le N$. For every prime power q, define*

$$D_q(Y_q, \ell) := \frac{\left|\left\{L \in \operatorname{Gr}(\mathbb{P}^\ell, \mathbb{P}^N)(\mathbb{F}_q) \;\middle|\; (L \cap Y_q)(\mathbb{F}_q) = \emptyset\right\}\right|}{|\operatorname{Gr}(\mathbb{P}^\ell, \mathbb{P}^N)(\mathbb{F}_q)|}.$$

(1) *There exists a constant c_1, depending only on $\mathscr{Y}$ and ℓ, such that, for every prime power q,*
$$0 \le D_q(Y_q, \ell) \le c_1 q^{N-d-\ell}.$$
In particular, if $d + \ell > N$, then
$$\lim_{q\to\infty} D_q(Y_q, \ell) = 0.$$

(2) *There exists a constant c_2, depending only on $\mathscr{Y}$ and ℓ, such that, for every prime power q,*
$$1 - c_2 q^{d+\ell-N} \le D_q(Y_q, \ell) \le 1.$$
In particular, if $d + \ell < N$, then
$$\lim_{q\to\infty} D_q(Y_q, \ell) = 1.$$

(3) *Suppose that $\Delta \geq 2$, $d+\ell = N$, and that every Y_q is quasireflexive. Then*
$$\lim_{q\to\infty} D_q(Y_q,\ell) = \sum_{i=0}^{\Delta} \frac{(-1)^i}{i!}.$$

For the application to rank-metric codes, fix m, n, and δ, and let
$$\mathscr{Y}_\delta \subseteq \mathbb{P}^{mn-1}_{\mathbb{Z}}$$
be the projective determinantal scheme cut out by the $\delta \times \delta$ minors of the generic $m \times n$ matrix. These minors have integer coefficients, so this defines a projective scheme over $\mathbb{Z}$. For every prime power q, its base change
$$Y_{\delta,q} := \mathscr{Y}_\delta \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Spec}(\mathbb{F}_q) \subseteq \mathbb{P}^{mn-1}_{\mathbb{F}_q}$$
is the determinantal variety of $m \times n$ matrices over $\mathbb{F}_q$ of rank less than δ.

The dimension and degree of these fibers are independent of q, and we prove in Theorem 4.13 that they are quasireflexive in every characteristic. We therefore apply Theorem 1.5 to $\mathscr{Y}_\delta$ with
$$N = mn-1, \qquad \ell = k-1.$$
The degree formula for determinantal varieties (see [Ful98, Example 14.4.11]) gives
$$\Delta = \deg(Y_{\delta,q}) = \prod_{i=0}^{n-\delta} \frac{\binom{m+i}{\delta-1}}{\binom{\delta-1+i}{\delta-1}}.$$
Moreover,
$$D_q(m\times n,k,\delta) = D_q(Y_{\delta,q},k-1).$$
Thus, Theorem 1.5 directly yields Corollary 1.3.

1.2. **Related work.** We now discuss how our method relates to previous approaches to density questions for rank-metric codes. Byrne and Ravagnani developed a combinatorial approach based on families of codes that are balanced with respect to a given partition of the ambient space [BR20]. For MRD codes with $n=\delta=2$, Antrobus and Gluesing-Luerssen utilized enumerative results on square matrices over $\mathbb{F}_q$ with no eigenvalue in $\mathbb{F}_q$ [AGL19]. Gluesing-Luerssen deduced an exact formula for $D_q(3\times 3,3,3)$ via the connection between square full-rank MRD codes and the theory of semifields [GL20, Theorem 2.4]. This connection was further developed by Gruica, Ravagnani, Sheekey, and Zullo in [GRSZ23, Section 3].

Gruica and Ravagnani [GR22] study density questions by counting common complements of collections of linear subspaces. As explained in [GKR23, Remark 6.6 (i)], every matrix $M \in \mathbb{F}_q^{mn}$ with $\operatorname{rank}(M) < \delta$ has row space contained in some $(\delta-1)$-dimensional subspace $U \subseteq \mathbb{F}_q^n$. For each such subspace, define
$$\mathbb{F}_q^{mn}(U) := \{M \in \operatorname{Mat}_{m\times n}(\mathbb{F}_q) \mid \operatorname{rowsp}(M) \subseteq U\}.$$
This is a linear subspace of $\mathbb{F}_q^{mn}$ of dimension $m(\delta-1)$. A code $C \subseteq \mathbb{F}_q^{mn}$ of dimension $m(n-\delta+1)$ has minimum rank distance at least δ if and only if $C \cap \mathbb{F}_q^{mn}(U) = \{0\}$ for every $(\delta-1)$-dimensional subspace $U \subseteq \mathbb{F}_q^n$. Since
$$\dim C + \dim \mathbb{F}_q^{mn}(U) = m(n-\delta+1) + m(\delta-1) = mn,$$
this condition is equivalent to
$$\mathbb{F}_q^{mn} = C \oplus \mathbb{F}_q^{mn}(U)$$

for every such U. Thus, the MRD codes of dimension $m(n-\delta+1)$ and minimum rank distance δ correspond to common complements of the spaces $\mathbb{F}_q^{mn}(U)$. The problem of estimating the number of such common complements can then be turned into one of counting isolated vertices in a certain bipartite graph. This approach fits into the study of the Critical Problem for combinatorial geometries. For a more extensive discussion of this connection, see [GRSZ23, Section 5].

In some ways, our geometric approach using linear sections of determinantal varieties is similar to the common-complement approach described above. The key distinction is that, rather than viewing the set of $\mathbb{F}_q$-matrices of rank at most $\delta-1$ as a union of finitely many linear subspaces, we work with the associated irreducible projective variety, exploiting more of the geometry inherent in the problem.

The previous work most closely related to our approach concerns a different kind of limit. In [NHTRR18], Neri, Horlemann-Trautmann, Randrianarisoa, and Rosenthal study the proportion of $\mathbb{F}_{q^s}$-linear MRD codes as the field extension degree s tends to infinity. Let $C \subseteq \mathbb{F}_{q^s}^n$ be an $\mathbb{F}_{q^s}$-linear MRD code of dimension k. After elementary row operations on a generator matrix and a permutation of the n coordinates, such a code admits a generator matrix in *systematic form*

$$G = (I_k \mid X).$$

Building on a general criterion for generator matrices of MRD codes [NHTRR18, Proposition 13], the authors show that the $k\times(n-k)$ matrices X yielding an MRD code form a Zariski open subset of the affine space of all such matrices [NHTRR18, Theorem 17]. Whereas their parameter space is an affine space of matrices, our parameter space is the Grassmannian of k-dimensional subspaces of the ambient matrix space, allowing us to exploit its intrinsic projective geometry.

We end the introduction with several directions not pursued in this paper:

Rate of convergence. Corollary 1.3 computes the limit of $D_q(m\times n, k, \delta)$ as $q\to\infty$, but it does not address the rate of convergence. As an illustrative example, in the case where $n=m=k=\delta=3$, our arguments yield an upper bound of the form $O(q^{-1})$ for the density, whereas Gluesing-Luerssen, using connections to the theory of semifields, obtained an exact formula that is $O(q^{-3})$ [GL20, Theorem 2.4]. Although Theorem 1.5 provides some information about the rate of convergence, we have not attempted to optimize it here.

Field of linearity. After fixing a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$, each element of $\mathbb{F}_{q^m}$ can be identified with a vector in $\mathbb{F}_q^m$, and hence each element of $\mathbb{F}_{q^m}^n$ with an $m\times n$ matrix over $\mathbb{F}_q$. Thus, if ℓ divides m, an $\mathbb{F}_{q^\ell}$-linear subspace of $\mathbb{F}_{q^m}^n$ corresponds to an $\mathbb{F}_q$-linear subspace of the space of $m\times n$ matrices. Note that every $\mathbb{F}_{q^\ell}$-linear code is $\mathbb{F}_q$-linear, whereas the converse does not hold in general. This issue is discussed explicitly in [GHRW24, Remark 5.2].

There has been considerable interest in density questions for such $\mathbb{F}_{q^\ell}$-linear rank-metric codes. In [GHRW24], Gruica, Horlemann, Ravagnani, and Willenborg investigate these questions in a broader setting. In particular, they consider an analogue of $D_q(m\times n, k, \delta)$ with an additional parameter specifying that only $\mathbb{F}_{q^\ell}$-linear subspaces are counted. They also study analogous density questions for codes endowed with other metrics, including the sum-rank metric.

Other limits of interest. In [NHTRR18], Neri, Horlemann-Trautmann, Randrianarisoa, and Rosenthal focus on the $\mathbb{F}_{q^m}$-linear rank-metric codes described above and show that $\mathbb{F}_{q^m}$-linear MRD codes are dense as $m\to\infty$. This result was generalized by Gruica, Horlemann, Ravagnani, and Willenborg, who proved that $\mathbb{F}_{q^\ell}$-linear MRD codes in $\mathbb{F}^n_{q^{\ell s}}$ are dense as $\ell \to \infty$ [GHRW24, Theorem 5.8].

Our geometric arguments, by contrast, require the full set of k-dimensional $\mathbb{F}_q$-linear subspaces of the space of $m \times n$ matrices over $\mathbb{F}_q$, rather than only those that are linear over some extension field $\mathbb{F}_{q^\ell}$. We also require the dimension of the matrix space to remain fixed as $q \to \infty$. Consequently, our results do not address other limits of interest, such as the limit of $D_q(m \times n, k, \delta)$ as $m \to \infty$.

Organization of the paper. The first two cases of Theorem 1.5 are proved in Section 2, while the remaining threshold case is treated in Section 3. In Section 4, we verify that determinantal varieties are quasireflexive. In Appendix A, we discuss the closely related notion of reflexivity, and prove that determinantal varieties satisfy this property in arbitrary characteristic.

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AI disclosure. The main results and the overall proof strategy were developed by the authors, who completed a draft of the paper before using any AI tools. ChatGPT (models 5.5 Thinking and 5.6 Sol) was subsequently used during the revision process. It assisted in reviewing mathematical arguments, identifying errors in the draft, suggesting corrections and proof strategies, and improving the clarity and presentation of the manuscript. For example, the proofs of Lemmas 4.6 and 4.7 are based on suggestions from ChatGPT. The authors carefully reviewed all AI-generated suggestions, independently verified the mathematical arguments and references, and made suitable adjustments to the final wording. The authors take full responsibility for the content of the paper.

2. Asymptotic density in the sparse and abundant cases

We prove the sparse and abundant cases of Theorem 1.5 in this section. For each prime power q, we write $Y = Y_q$. All implied constants below are uniform in q. Indeed, the fibers Y_q, the associated incidence correspondences, and their projection images have complexity bounded independently of q, as they arise from the fixed equations defining $\mathscr{Y} \subseteq \mathbb{P}^N_{\mathbb{Z}}$. We use point-counting estimates in their bounded-complexity form; see [Ent21, Section 4] and Remark 3.2 for more details.

2.1. **The sparse case.** For a projective variety $Y \subseteq \mathbb{P}^N_{\mathbb{F}_q}$, choose $L \in \mathrm{Gr}(\mathbb{P}^\ell, \mathbb{P}^N)(\mathbb{F}_q)$ uniformly at random and define

$$\begin{aligned} \Theta\colon\ \mathrm{Gr}(\mathbb{P}^\ell, \mathbb{P}^N)(\mathbb{F}_q) &\longrightarrow \mathbb{Z}_{\geq 0} \\ L &\longmapsto |(L \cap Y)(\mathbb{F}_q)|. \end{aligned}$$

We view Θ as a random variable. As shown for hyperplanes in [PS22, Lemma 4.1] and for higher-codimensional linear sections in [KS22, Lemma 2.1], the mean μ and variance σ^2 of Θ satisfy

$$\mu = |Y(\mathbb{F}_q)| \left(q^{\ell-N} + O\left(q^{\ell-N-1}\right)\right), \qquad \sigma^2 = O\left(q^{\ell-N}|Y(\mathbb{F}_q)|\right).$$

If Y is geometrically irreducible, the Lang–Weil bound [LW54, Theorem 1] gives

$$|Y(\mathbb{F}_q)| = q^{\dim Y} + O\left(q^{\dim Y - \frac{1}{2}}\right).$$

Substituting this into the preceding estimates, we obtain

$$\mu = q^{\dim Y + \ell - N} + O\left(q^{\dim Y + \ell - N - \frac{1}{2}}\right), \qquad \sigma^2 = O\left(q^{\dim Y + \ell - N}\right). \tag{2.1}$$

Proof of Theorem 1.5 (1). The inequality $D_q(Y,\ell) \geq 0$ holds by definition, so it remains to show that there exists a constant c_1, depending only on the fixed scheme $\mathscr{Y}$ and on ℓ such that

$$D_q(Y,\ell) \leq c_1 q^{N - \dim Y - \ell}.$$

Consider the set of pointless linear sections

$$\mathcal{U}_q := \left\{ L \in \mathrm{Gr}(\mathbb{P}^\ell, \mathbb{P}^N)(\mathbb{F}_q) \;\middle|\; (L \cap Y)(\mathbb{F}_q) = \emptyset \right\}.$$

For all sufficiently large q, Equation (2.1) gives $\mu > 0$. Since $\Theta = 0$ on $\mathcal{U}_q$, Chebyshev's inequality then gives

$$\begin{aligned} D_q(Y,\ell) = \mathrm{Prob}(\Theta = 0) &\leq \mathrm{Prob}(|\Theta - \mu| \geq \mu) \\ &\leq \frac{\sigma^2}{\mu^2} = O\left(q^{N - \dim Y - \ell}\right). \end{aligned}$$

If $\sigma = 0$, then $\Theta \equiv \mu > 0$, so $\mathcal{U}_q$ is empty and the bound holds trivially. Increasing the constant to cover the finitely many smaller prime powers establishes the desired upper bound for every q. In particular, the condition $\dim Y + \ell > N$ implies $\lim_{q\to\infty} q^{N-\dim Y-\ell} = 0$, which further implies $\lim_{q\to\infty} D_q(Y,\ell) = 0$. $\square$

2.2. **The abundant case.** Consider a geometrically irreducible variety $Y \subseteq \mathbb{P}^N$ defined over $\mathbb{F}_q$, together with the incidence correspondence

$$\mathcal{X} := \left\{ (y, L) \in Y \times \mathrm{Gr}(\mathbb{P}^\ell, \mathbb{P}^N) \;\middle|\; y \in L \right\}.$$

There are two natural projection maps

$$\begin{tikzcd} & \mathcal{X} \arrow[dl, "\pi_Y"'] \arrow[dr, "\pi_{\mathrm{Gr}}"] & \\ Y & & \mathrm{Gr}(\mathbb{P}^\ell, \mathbb{P}^N), \end{tikzcd}$$

where the image of the second projection is

$$\mathcal{N} := \pi_{\mathrm{Gr}}(\mathcal{X}) = \left\{ L \in \mathrm{Gr}(\mathbb{P}^\ell, \mathbb{P}^N) \;\middle|\; L \cap Y \neq \emptyset \right\}.$$

To clarify the (geometric) points of $\mathcal{N}$, observe that

$$\mathcal{N}(\overline{\mathbb{F}_q}) = \left\{ L \in \mathrm{Gr}(\mathbb{P}^\ell, \mathbb{P}^N)(\overline{\mathbb{F}_q}) \;\middle|\; L \cap Y_{\overline{\mathbb{F}_q}} \neq \emptyset \right\}. \tag{2.2}$$

Lemma 2.1. *The incidence correspondence $\mathcal{X}$ is geometrically irreducible with*

$$\dim \mathcal{X} = \dim \mathrm{Gr}(\mathbb{P}^\ell, \mathbb{P}^N) - (N - \dim Y - \ell).$$

Proof. If $\ell = 0$, then $\mathrm{Gr}(\mathbb{P}^0, \mathbb{P}^N) \cong \mathbb{P}^N$, and $\mathcal{X}$ is the graph of the inclusion $Y \hookrightarrow \mathbb{P}^N$. Since $\mathcal{X} \cong Y$, the assertion follows. For the remainder of the proof, we assume that $\ell \geq 1$.

For any $y \in Y$, the fiber $\pi_Y^{-1}(y)$ consists of ℓ-dimensional linear subspaces of $\mathbb{P}^N$ containing y. Projecting from y identifies such subspaces with $(\ell-1)$-dimensional linear subspaces of $\mathbb{P}^{N-1}$, yielding an isomorphism

$$\pi_Y^{-1}(y) \cong \mathrm{Gr}(\mathbb{P}^{\ell-1}, \mathbb{P}^{N-1}).$$

Hence, the fibers of π_Y are geometrically irreducible and have the same dimension. Since Y is geometrically irreducible, it follows that $\mathcal{X}$ is geometrically irreducible by [Sha13, Theorem 1.26].

For the dimension, we compute

$$\begin{aligned}
\dim \mathcal{X} = \dim \pi_Y^{-1}(y) + \dim Y &= \dim \mathrm{Gr}(\mathbb{P}^{\ell-1}, \mathbb{P}^{N-1}) + \dim Y \\
&= \ell(N-\ell) + \dim Y \\
&= \ell(N-\ell) + (N-\ell) + \dim Y - (N-\ell) \\
&= (\ell+1)(N-\ell) + \dim Y - (N-\ell) \\
&= \dim \mathrm{Gr}(\mathbb{P}^{\ell}, \mathbb{P}^{N}) - (N - \dim Y - \ell).
\end{aligned}$$

This proves the statement. □

Lemma 2.2. *The projection image $\mathcal{N} = \pi_{\mathrm{Gr}}(\mathcal{X})$ is geometrically irreducible with*

$$\dim \mathcal{N} \leq \dim \mathrm{Gr}(\mathbb{P}^{\ell}, \mathbb{P}^{N}) - (N - \dim Y - \ell).$$

Proof. The variety $\mathcal{N}$ is geometrically irreducible since $\mathcal{X}$ is geometrically irreducible by Lemma 2.1. The upper bound on the dimension follows from the inequality $\dim \mathcal{N} \leq \dim \mathcal{X}$ together with Lemma 2.1. □

Proof of Theorem 1.5 (2). By definition,

$$1 - D_q(Y, \ell) = \frac{\left| \left\{ L \in \mathrm{Gr}(\mathbb{P}^{\ell}, \mathbb{P}^{N})(\mathbb{F}_q) \;\middle|\; (L \cap Y)(\mathbb{F}_q) \neq \emptyset \right\} \right|}{|\,\mathrm{Gr}(\mathbb{P}^{\ell}, \mathbb{P}^{N})(\mathbb{F}_q)|}.$$

In view of (2.2), observe that $\left\{ L \in \mathrm{Gr}(\mathbb{P}^{\ell}, \mathbb{P}^{N})(\mathbb{F}_q) \;\middle|\; (L \cap Y)(\mathbb{F}_q) \neq \emptyset \right\}$ is a subset of $\mathcal{N}(\mathbb{F}_q)$. This inclusion may be strict, since an $\mathbb{F}_q$-rational linear subspace can meet Y geometrically without meeting $Y(\mathbb{F}_q)$. In any event,

$$1 - D_q(Y, \ell) \leq \frac{|\mathcal{N}(\mathbb{F}_q)|}{|\,\mathrm{Gr}(\mathbb{P}^{\ell}, \mathbb{P}^{N})(\mathbb{F}_q)|}. \tag{2.3}$$

The variety $\mathcal{N}$ is geometrically irreducible by Lemma 2.2. By the bounded-complexity form of the Lang–Weil estimate [CGH23, Theorem 4.6], there is a constant A, independent of q, such that

$$\left| |\mathcal{N}(\mathbb{F}_q)| - q^{\dim \mathcal{N}} \right| \leq A q^{\dim \mathcal{N} - 1/2}.$$

From this inequality, one can check that for $q \geq \max\{A^2, 1\}$,

$$|\mathcal{N}(\mathbb{F}_q)| \leq q^{\dim \mathcal{N}} + A q^{\dim \mathcal{N} - 1/2} \leq 2 q^{\dim \mathcal{N}}. \tag{2.4}$$

Next, we observe that

$$|\,\mathrm{Gr}(\mathbb{P}^{\ell}, \mathbb{P}^{N})(\mathbb{F}_q)| = \prod_{j=0}^{\ell} \frac{q^{N+1-j} - 1}{q^{\ell+1-j} - 1} \geq q^{(\ell+1)(N-\ell)} = q^{\dim \mathrm{Gr}(\mathbb{P}^{\ell}, \mathbb{P}^{N})}. \tag{2.5}$$

Indeed, each factor is at least $q^{N-\ell}$, since $(q^{N+1-j}-1) - q^{N-\ell}(q^{\ell+1-j}-1) = q^{N-\ell}-1 \geq 0$.

Applying Lemma 2.2 in combination with (2.3), (2.4), and (2.5), we obtain the following estimate for all sufficiently large q:

$$1 - D_q(Y,\ell) \leq \frac{2q^{\dim \mathrm{Gr}(\mathbb{P}^\ell,\mathbb{P}^N)-(N-\dim Y-\ell)}}{q^{\dim \mathrm{Gr}(\mathbb{P}^\ell,\mathbb{P}^N)}} = 2q^{\dim Y+\ell-N}.$$

Increasing the constant to cover the finitely many smaller prime powers, we obtain a constant c_2 depending only on $\mathscr{Y}$ and ℓ such that, for every q,

$$1 - D_q(Y,\ell) \leq c_2 q^{\dim Y+\ell-N}.$$

Rearranging the terms gives

$$1 - c_2 q^{\dim Y+\ell-N} \leq D_q(Y,\ell) \leq 1.$$

If $\dim Y + \ell < N$, then $\lim_{q\to\infty}(1 - c_2 q^{\dim Y+\ell-N}) = 1$, and thus $\lim_{q\to\infty} D_q(Y,\ell) = 1$. $\square$

3. Asymptotic density in the threshold case

Consider a geometrically irreducible variety $Y \subseteq \mathbb{P}^N$ of degree Δ over $\mathbb{F}_q$, together with the incidence correspondence

$$\begin{tikzcd}[column sep=small] & \mathcal{X} := \left\{(y,L) \in Y \times \mathrm{Gr}(\mathbb{P}^\ell,\mathbb{P}^N) \;\middle|\; y \in L\right\} \arrow[dl, "\pi_Y"'] \arrow[dr, "\pi_{\mathrm{Gr}}"] & \\ Y & & \mathcal{G} := \mathrm{Gr}(\mathbb{P}^\ell,\mathbb{P}^N). \end{tikzcd}$$

Assume that $\dim Y + \ell = N$. Then a general ℓ-plane $L \in \mathcal{G}$ intersects Y transversely in Δ points by Bertini's theorem. In particular, the map

$$\pi_{\mathrm{Gr}} \colon \mathcal{X} \longrightarrow \mathcal{G}$$

is generically finite of degree Δ. Let $\mathcal{G}^\circ \subseteq \mathcal{G}$ be the locus of ℓ-planes whose intersection with Y consists of Δ distinct geometric points, all smooth on Y. Observe that $\mathcal{G}^\circ$ is a nonempty Zariski open subset over which π_{Gr} is finite étale of degree Δ. In this setting, Theorem 1.5 (3) concerns the distribution of $\mathbb{F}_q$-points $L \in \mathcal{G}(\mathbb{F}_q)$ for which the fiber $\pi_{\mathrm{Gr}}^{-1}(L) = L \cap Y$ contains no $\mathbb{F}_q$-points.

3.1. **Chebotarev density theorem.** Before proceeding to the proof, we briefly review the background for the key ingredient. The main source for the following material is [Ent21, Sections 3 and 4]. Let $\mathcal{G}^\circ$ be a geometrically irreducible quasi-projective variety, and let

$$f \colon \mathcal{X}^\circ \longrightarrow \mathcal{G}^\circ$$

be a finite étale morphism of degree Δ. For a geometric point $L_0 \in \mathcal{G}^\circ(\overline{\mathbb{F}_q})$, the étale fundamental group $\pi^{\mathrm{et}}(\mathcal{G}^\circ, L_0)$ acts on the fiber $f^{-1}(L_0)$, giving rise to a homomorphism to the symmetric group S_Δ. The *arithmetic monodromy group* $\mathrm{Mon}(\mathcal{X}^\circ/\mathcal{G}^\circ)$ is defined as the image of this homomorphism. This group is well defined up to conjugation since different choices of the base point L_0 yield conjugate subgroups of S_Δ.

On the other hand, the *geometric monodromy group* is defined by

$$\mathrm{Mon}^g(\mathcal{X}^\circ/\mathcal{G}^\circ) := \mathrm{Mon}\left(\mathcal{X}^\circ \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}/\mathcal{G}^\circ \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}\right).$$

The two monodromy groups fit into a short exact sequence

$$1 \longrightarrow \mathrm{Mon}^g(\mathcal{X}^\circ/\mathcal{G}^\circ) \longrightarrow \mathrm{Mon}(\mathcal{X}^\circ/\mathcal{G}^\circ) \longrightarrow \mathrm{Gal}(\mathbb{F}_{q^\nu}/\mathbb{F}_q) \longrightarrow 1,$$

where ν is the minimal positive integer such that

$$\mathrm{Mon}^g(\mathcal{X}^\circ/\mathcal{G}^\circ) = \mathrm{Mon}\left(\mathcal{X}^\circ \times_{\mathbb{F}_q} \mathbb{F}_{q^\nu}/\mathcal{G}^\circ \times_{\mathbb{F}_q} \mathbb{F}_{q^\nu}\right).$$

In particular, the arithmetic and geometric monodromy groups may not coincide, even when q is large. In our application, we will prove directly that the geometric monodromy group is S_Δ. Since the arithmetic monodromy group is a subgroup of S_Δ containing the geometric monodromy group, it will follow that both groups are equal to S_Δ.

For each point $L \in \mathcal{G}^\circ(\mathbb{F}_q)$, the fiber $f^{-1}(L) \subseteq \mathcal{X}^\circ$ is a 0-dimensional scheme over $\mathbb{F}_q$ consisting of Δ geometric points. The action of the Frobenius map on this fiber determines a conjugacy class

$$\mathrm{Frob}_q(L) \subseteq \mathrm{Mon}(\mathcal{X}^\circ/\mathcal{G}^\circ).$$

Note that $f^{-1}(L)$ contains no $\mathbb{F}_q$-points if and only if the class $\mathrm{Frob}_q(L)$ consists of derangements. This idea reduces our problem to estimating the density of points $L \in \mathcal{G}^\circ(\mathbb{F}_q)$ for which $\mathrm{Frob}_q(L)$ consists of derangements.

The key tool for our argument is the following version of the Chebotarev density theorem proved by Entin [Ent21]. We state only the form relevant to our setting, namely the case in which the arithmetic and geometric monodromy groups coincide.

Theorem 3.1 ([Ent21, Theorem 3])**.** *Let $f\colon \mathcal{X}^\circ \to \mathcal{G}^\circ$ be a finite étale morphism between quasiprojective varieties over $\mathbb{F}_q$ with $\mathcal{G}^\circ$ smooth and geometrically irreducible. Suppose that*

$$\mathrm{Mon}^g(\mathcal{X}^\circ/\mathcal{G}^\circ) = \mathrm{Mon}(\mathcal{X}^\circ/\mathcal{G}^\circ).$$

Then for each conjugacy class $\mathcal{C} \subseteq \mathrm{Mon}(\mathcal{X}^\circ/\mathcal{G}^\circ)$, we have

$$|\{L \in \mathcal{G}^\circ(\mathbb{F}_q) \mid \mathrm{Frob}_q(L) = \mathcal{C}\}| = \frac{|\mathcal{C}|}{|\,\mathrm{Mon}(\mathcal{X}^\circ/\mathcal{G}^\circ)|} q^{\dim \mathcal{G}^\circ}\left(1 + O\left(q^{-1/2}\right)\right).$$

Remark 3.2. The implied constant in $O\left(q^{-1/2}\right)$ depends on the *complexity* of the morphism f. Roughly speaking, this complexity measures the ambient projective dimension and the degrees of the equations and nonvanishing conditions defining the graph of f; see [Ent21, Section 4]. In our setting, $\mathscr{Y}$ is defined by fixed equations over $\mathbb{Z}$, and the incidence and transversality conditions have bounded complexity as q varies. The same holds for their images, the map f, and the complement $\mathcal{D} = \mathcal{G} \setminus \mathcal{G}^\circ$. It follows that the constants in the point-counting estimates and in the Chebotarev theorem depend only on $\mathscr{Y}$ and ℓ.

3.2. **Proof for the threshold case.** We retain the notation introduced at the beginning of this section and set $\mathcal{X}^\circ := \pi_{\mathrm{Gr}}^{-1}(\mathcal{G}^\circ)$. The restriction

$$f := \pi_{\mathrm{Gr}}|_{\mathcal{X}^\circ}\colon \mathcal{X}^\circ \longrightarrow \mathcal{G}^\circ$$

is therefore a finite étale morphism of degree Δ.

Lemma 3.3. *If the variety Y is quasireflexive, then* $\mathrm{Mon}^g(\mathcal{X}^\circ/\mathcal{G}^\circ) = \mathrm{Mon}(\mathcal{X}^\circ/\mathcal{G}^\circ) \cong S_\Delta$.

Proof. We first work over $\overline{\mathbb{F}_q}$. Quasireflexivity implies $\Delta \geq 2$ and $\ell \geq 1$. Fix a geometric point $L_0 \in \mathcal{G}^\circ(\overline{\mathbb{F}_q})$ and identify $\mathrm{Mon}^g(\mathcal{X}^\circ/\mathcal{G}^\circ)$ as a subgroup of S_Δ through its action on the geometric fiber $f^{-1}(L_0)$. Let

$$D \subseteq \mathcal{X} \times_{\mathcal{G}} \mathcal{X} \qquad \text{and} \qquad D' \subseteq Y \times Y$$

denote the diagonals. The fiber of the morphism

$$\mathcal{X} \times_{\mathcal{G}} \mathcal{X} \setminus D \longrightarrow Y \times Y \setminus D'$$

over $(y_1, y_2) \in Y \times Y \setminus D'$ consists of the points $((y_1, L), (y_2, L))$, where $L \in \mathcal{G} = \mathrm{Gr}(\mathbb{P}^\ell, \mathbb{P}^N)$ contains the line spanned by y_1 and y_2. This realizes $\mathcal{X} \times_{\mathcal{G}} \mathcal{X} \setminus D$ as a Grassmannian bundle over the irreducible variety $Y \times Y \setminus D'$, so it is itself irreducible. Its restriction

$$(\mathcal{X}^\circ \times_{\mathcal{G}^\circ} \mathcal{X}^\circ) \setminus D$$

is an open subset, nonempty since $\Delta \geq 2$, and hence irreducible. It follows that $\mathrm{Mon}^g(\mathcal{X}^\circ/\mathcal{G}^\circ)$ acts doubly transitively on $f^{-1}(L_0)$ by [BH86, Proposition 2 (ii)].

Quasireflexivity gives $L \in \mathcal{G}$ whose intersection with Y consists of one double point y_0 and $\Delta - 2$ transverse points, all smooth on Y. The projection map π_{Gr} is finite near L and étale at the transverse points. Since $\mathcal{X} \to Y$ is a Grassmannian bundle, $\mathcal{X}$ is smooth, hence formally irreducible, at (y_0, L); the base $\mathcal{G}$ is also smooth. By [BH86, Proposition 3], as restated in [Ent21, Proposition 3.1], the geometric monodromy group contains a transposition in every characteristic. Double transitivity now gives $\mathrm{Mon}^g(\mathcal{X}^\circ/\mathcal{G}^\circ) = S_\Delta$. Finally, the chain

$$S_\Delta = \mathrm{Mon}^g(\mathcal{X}^\circ/\mathcal{G}^\circ) \subseteq \mathrm{Mon}(\mathcal{X}^\circ/\mathcal{G}^\circ) \subseteq S_\Delta$$

shows that equality must hold throughout, as desired. $\square$

Lemma 3.4. *If the variety Y is quasireflexive, then, as $q \to \infty$,*

$$\left|\left\{L \in \mathcal{G}^\circ(\mathbb{F}_q) \;\middle|\; \mathrm{Frob}_q \text{ deranges } \pi_{\mathrm{Gr}}^{-1}(L)\right\}\right| = \left(\sum_{i=0}^{\Delta} \frac{(-1)^i}{i!}\right) q^{\dim \mathcal{G}} \left(1 + O\left(q^{-1/2}\right)\right).$$

Proof. By Lemma 3.3, the geometric and arithmetic monodromy groups are both the symmetric group S_Δ. Applying Theorem 3.1, we obtain

$$\begin{aligned}\left|\left\{L \in \mathcal{G}^\circ(\mathbb{F}_q) \;\middle|\; \mathrm{Frob}_q \text{ deranges } \pi_{\mathrm{Gr}}^{-1}(L)\right\}\right| &= \sum_{\substack{\mathcal{C} \text{ consists of} \\ \text{derangements}}} \frac{|\mathcal{C}|}{|S_\Delta|} q^{\dim \mathcal{G}} \left(1 + O\left(q^{-1/2}\right)\right) \\ &= \frac{|\{\text{derangements in } S_\Delta\}|}{|S_\Delta|} q^{\dim \mathcal{G}} \left(1 + O\left(q^{-1/2}\right)\right) \\ &= \left(\sum_{i=0}^{\Delta} \frac{(-1)^i}{i!}\right) q^{\dim \mathcal{G}} \left(1 + O\left(q^{-1/2}\right)\right),\end{aligned}$$

as claimed. $\square$

Proof of Theorem 1.5 (3). By definition,

$$D_q(Y, \ell) = \frac{|\{L \in \mathcal{G}(\mathbb{F}_q) \mid (L \cap Y)(\mathbb{F}_q) = \emptyset\}|}{|\mathcal{G}(\mathbb{F}_q)|}.$$

The condition $(L \cap Y)(\mathbb{F}_q) = \emptyset$ holds if and only if the Frobenius endomorphism fixes no point on the fiber $\pi_{\mathrm{Gr}}^{-1}(L)$. Therefore, over the open subset $\mathcal{G}^\circ \subseteq \mathcal{G}$, Lemma 3.4 yields

$$|\{L \in \mathcal{G}^\circ(\mathbb{F}_q) \mid (L \cap Y)(\mathbb{F}_q) = \emptyset\}| = \left(\sum_{i=0}^{\Delta} \frac{(-1)^i}{i!}\right) q^{\dim \mathcal{G}} \left(1 + O\left(q^{-1/2}\right)\right).$$

The complement $\mathcal{D} := \mathcal{G} \setminus \mathcal{G}^\circ$ is a proper closed subset, so $\dim \mathcal{D} < \dim \mathcal{G}$. Thus, by the Lang–Weil bound [LW54, Lemma 1],

$$|\mathcal{D}(\mathbb{F}_q)| \leq O\left(q^{\dim \mathcal{G} - 1}\right).$$

Comparing the two estimates, the contribution from $\mathcal{D}(\mathbb{F}_q)$ is negligible. On the other hand, the denominator satisfies $|\mathcal{G}(\mathbb{F}_q)| = q^{\dim \mathcal{G}}\left(1 + O\left(q^{-1}\right)\right)$. Therefore,

$$D_q(Y, \ell) = \sum_{i=0}^{\Delta} \frac{(-1)^i}{i!} + O\left(q^{-1/2}\right).$$

In particular,

$$\lim_{q\to\infty} D_q(Y, \ell) = \sum_{i=0}^{\Delta} \frac{(-1)^i}{i!}.$$

This completes the proof. □

4. Quasireflexivity of determinantal varieties

In this section, we prove that determinantal varieties are quasireflexive over fields of arbitrary characteristic. Throughout, we fix the following notation. Let K be an algebraically closed field and fix integers

$$2 \le \delta \le n \le m.$$

We consider the determinantal variety

$$Y_\delta \subseteq \mathbb{P}^N \qquad \text{where} \qquad N = mn - 1$$

consisting of $m \times n$ matrices of rank less than δ. To simplify the notation, we denote its dimension and codimension by

$$\begin{aligned} d &:= \dim Y_\delta = N - (m - \delta + 1)(n - \delta + 1), \\ c &:= \operatorname{codim} Y_\delta = (m - \delta + 1)(n - \delta + 1). \end{aligned}$$

By Definition 1.4, to show that Y_δ is quasireflexive, we need to find a c-dimensional linear subspace $L \subseteq \mathbb{P}^N$ that intersects Y_δ in $\deg(Y_\delta) - 1$ points all lying in the smooth locus

$$Y_\delta^{\mathrm{sm}} \subseteq Y_\delta.$$

We begin by treating the hypersurface case.

Proposition 4.1. *If Y_δ is a hypersurface, that is, if $m = n = \delta$, then it is quasireflexive.*

Proof. In this setting, $Y_\delta \subseteq \mathbb{P}^N = \mathbb{P}^{n^2-1}$ is a hypersurface of degree n. It suffices to construct a line $L \subseteq \mathbb{P}^N$ meeting the smooth locus Y_δ^{sm} in exactly $n-1$ distinct points.

Choose $n - 1$ distinct points $[a_i : b_i] \in \mathbb{P}^1$, where $i = 1, \ldots, n-1$. When $n = 2$, choose in addition a point $[a_2 : b_2] \ne [a_1 : b_1]$. Consider the morphism

$$\varphi\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^N, \quad \varphi(s:t) = \begin{pmatrix} a_1 s + b_1 t & a_2 s + b_2 t & 0 & \cdots & 0 \\ 0 & a_1 s + b_1 t & 0 & \cdots & 0 \\ 0 & 0 & a_2 s + b_2 t & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} s + b_{n-1} t \end{pmatrix}.$$

When $n = 2$, this matrix is interpreted as

$$\begin{pmatrix} a_1 s + b_1 t & a_2 s + b_2 t \\ 0 & a_1 s + b_1 t \end{pmatrix}.$$

Let $L := \varphi(\mathbb{P}^1)$, which is a line in $\mathbb{P}^N$. The scheme-theoretic intersection $L \cap Y_\delta$ is defined by the equation $\det(\varphi(s:t)) = 0$, namely

$$(a_1 s + b_1 t)^2 (a_2 s + b_2 t) \cdots (a_{n-1} s + b_{n-1} t) = 0.$$

When $n = 2$, this equation reduces to $(a_1 s + b_1 t)^2 = 0$. The determinant equation above has a double root at $[b_1 : -a_1]$ and $n-2$ simple roots at $[b_i : -a_i]$ for $2 \le i \le n-1$.

For the matrix $\varphi(b_1 : -a_1)$, the first two diagonal entries vanish, while the $(1,2)$-entry equals $a_2 b_1 - a_1 b_2 \neq 0$, and all remaining diagonal entries are nonzero. As the matrix has rank $n-1$, it represents a point in the smooth locus Y_δ^{sm}. At each remaining root $[b_i : -a_i]$, where $i \ge 2$, exactly one diagonal entry vanishes, while the leading 2×2 block is invertible. Thus the matrix again has rank $n-1$, so every remaining intersection point also lies in Y_δ^{sm}. Therefore, L meets Y_δ^{sm} in $n-1$ distinct points. This proves that Y_δ is quasireflexive. $\square$

4.1. **Preliminary lemmas on tangent spaces.** The remainder of this section is devoted to the non-hypersurface case. Here, we collect properties of the tangent spaces of a determinantal variety that are needed for our proof.

For any point $P \in Y_\delta^{\mathrm{sm}}$, we write its embedded tangent space as $T_P Y_\delta \subseteq \mathbb{P}^N$. Recall that, as shown for example in [Har92, Example 14.16],

$$T_P Y_\delta = \left\{ Q \in \mathbb{P}^N \;\middle|\; Q(\ker P) \subseteq \operatorname{im} P \right\}.$$

We will make use of the distinguished smooth point

$$P_0 = \begin{pmatrix} I_{\delta-1} & 0 \\ 0 & 0 \end{pmatrix} \in Y_\delta^{\mathrm{sm}},$$

where $I_{\delta-1}$ denotes the $(\delta-1) \times (\delta-1)$ identity matrix. Its tangent space $T_{P_0} Y_\delta$ consists of points represented by matrices of the form

$$\begin{pmatrix} *_{\delta-1,\,\delta-1} & *_{\delta-1,\,n-\delta+1} \\ *_{m-\delta+1,\,\delta-1} & 0 \end{pmatrix} \tag{4.1}$$

where $*_{\alpha,\,\beta}$ denotes an arbitrary $\alpha \times \beta$ matrix.

We begin by analyzing the structure of the linear section $Y_\delta^{\mathrm{sm}} \cap T_{P_0} Y_\delta$. This section admits a stratification according to the dimension of the intersection $\ker P_0 \cap \ker P \subseteq K^n$. More precisely, we have

$$Y_\delta^{\mathrm{sm}} \cap T_{P_0} Y_\delta = \bigcup_{i=0}^{n-\delta+1} T_i$$

where

$$T_i = \{ P \in Y_\delta^{\mathrm{sm}} \cap T_{P_0} Y_\delta \mid \dim(\ker P_0 \cap \ker P) = (n-\delta+1) - i \}.$$

In particular, since $\ker P_0$ has dimension $n - \delta + 1$, we have

$$T_0 = \{ P \in Y_\delta^{\mathrm{sm}} \cap T_{P_0} Y_\delta \mid \ker P_0 = \ker P \}.$$

Lemma 4.2. *The Zariski closure $\overline{T_0} \subseteq Y_\delta$ is a linear space containing P_0. The stratum T_i is nonempty if and only if $0 \le i \le \min\{\delta - 1, n - \delta + 1\}$. Each nonempty T_i has positive codimension in Y_δ^{sm}. If Y_δ is not a hypersurface, only T_0 can have codimension one, and this occurs if and only if $\delta = n = 2$.*

Proof. The points of T_0 are precisely the rank-$(\delta-1)$ matrices in the linear space

$$\left\{P \in \mathbb{P}^N \;\middle|\; P(\ker P_0) = 0\right\}.$$

This linear space is contained in Y_δ and contains P_0. Since T_0 is a nonempty open subset of it, $\overline{T_0}$ is the whole linear space.

To compute the dimension of T_i, we first decompose

$$K^n \cong \ker P_0 \oplus K^{\delta-1}.$$

For each $P \in T_i$, the restrictions of a linear map $\widetilde{P}\colon K^n \longrightarrow K^m$ representing it to the two summands behave as follows:

- Since $P \in T_{P_0}Y_\delta$, the restriction to $\ker P_0$ defines a linear map
$$A\colon\ \ker P_0 \cong K^{n-\delta+1} \longrightarrow \operatorname{im} P_0 \cong K^{\delta-1}$$
of rank i.
- The restriction to the second summand defines a linear map
$$B\colon K^{\delta-1} \longrightarrow K^m.$$
Its composition with the quotient map gives
$$\overline{B}\colon K^{\delta-1} \longrightarrow K^m/\operatorname{im} A.$$
Since $\operatorname{rank} \widetilde{P} = i + \operatorname{rank} \overline{B} = \delta - 1$, the map $\overline{B}$ has rank $\delta - 1 - i$.

Conversely, any pair of restrictions satisfying these conditions determines a point of T_i. Such choices exist precisely when

$$0 \le i \le \min\{\delta-1, n-\delta+1\}.$$

The rank-i maps A form a variety of dimension $i(n-i)$. For fixed A, the maps $\overline{B}$ of rank $\delta-1-i$ form a variety of dimension $(\delta-1-i)\big((\delta-1)+(m-i)-(\delta-1-i)\big) = m(\delta-1-i)$. Each such map has an affine space of lifts B of dimension

$$\dim \operatorname{Hom}\left(K^{\delta-1}, \operatorname{im} A\right) = i(\delta-1).$$

Subtracting one for projectivization, we obtain

$$\dim T_i = i(n-i) + m(\delta-1-i) + i(\delta-1) - 1.$$

To determine the codimension of T_i in Y_δ^{sm}, we compute

$$\begin{aligned}\dim Y_\delta - \dim T_i &= (\delta-1)(m+n-\delta+1) - \big(i(n-i) + m(\delta-1-i) + i(\delta-1)\big)\\ &= (\delta-1-i)(n-\delta+1-i) + i(m-\delta+1).\end{aligned}$$

For $i = 0$, this is $(\delta-1)(n-\delta+1)$, which is positive and equals one if and only if $\delta = n = 2$. For $i > 0$, the codimension is at least $i(m-\delta+1) > 0$. If it equals one, then $i = 1$ and $m = \delta$; since $\delta \le n \le m$, this forces $m = n = \delta$. Thus, outside the hypersurface case, only T_0 can have codimension one. This completes the proof. $\square$

Lemma 4.3. *For a general $P \in Y_\delta^{\mathrm{sm}}$, the tangent space T_PY_δ does not contain P_0.*

Proof. Since Y_δ is irreducible and the locus of $P \in Y_\delta^{\mathrm{sm}}$ for which T_PY_δ contains P_0 is Zariski closed, it suffices to find a point P such that T_PY_δ does not contain P_0. Let $P \in Y_\delta^{\mathrm{sm}}$ be the point represented by the matrix whose (i,i)-entries are equal to 1 for $i = 2, \ldots, \delta$, and zero elsewhere. Let e_i denote the ith standard basis vector in K^n or K^m, as appropriate. Then

$$e_1 \in \ker P \quad \text{and} \quad \operatorname{im} P = \operatorname{span}(e_2, \ldots, e_\delta).$$

Since $P_0(e_1) = e_1$, we have

$$P_0(\ker P) \nsubseteq \operatorname{im} P,$$

and hence $P_0 \notin T_P Y_\delta$. This gives an example of P with the desired property. $\square$

Lemma 4.4. *If Y_δ is not a hypersurface, then for a general point $P \in Y_\delta^{\mathrm{sm}}$, we have*

$$\dim(T_{P_0} Y_\delta \cap T_P Y_\delta) \leq d - 2.$$

Proof. Since Y_δ is irreducible and the function $P \longmapsto \dim(T_{P_0} Y_\delta \cap T_P Y_\delta)$ is upper semicontinuous on Y_δ^{sm}, it suffices to find a point P with

$$\dim(T_{P_0} Y_\delta \cap T_P Y_\delta) \leq d - 2,$$

or equivalently, such that $T_{P_0} Y_\delta \cap T_P Y_\delta$ has codimension at least 2 in $T_{P_0} Y_\delta$. Consider

$$P = \begin{pmatrix} 0 & 0 \\ 0 & I_{\delta-1} \end{pmatrix} \in Y_\delta^{\mathrm{sm}}.$$

Its tangent space $T_P Y_\delta$ consists of points represented by matrices of the form

$$\begin{pmatrix} 0 & *_{m-\delta+1,\,\delta-1} \\ *_{\delta-1,\,n-\delta+1} & *_{\delta-1,\,\delta-1} \end{pmatrix}.$$

Comparing this with (4.1), we see that the intersection $T_{P_0} Y_\delta \cap T_P Y_\delta$ consists of points represented by $m \times n$ matrices with $(m - \delta + 1) \times (n - \delta + 1)$ zero matrices as the top-left and bottom-right blocks.

- Suppose that

  $$m - \delta + 1 \leq \delta - 1 \qquad \text{or} \qquad n - \delta + 1 \leq \delta - 1.$$

  In this case, the two zero blocks do not overlap. Thus the linear subspace

  $$T_{P_0} Y_\delta \cap T_P Y_\delta \subseteq T_{P_0} Y_\delta$$

  is defined by $(m - \delta + 1)(n - \delta + 1)$ independent linear conditions. Hence

  $$\operatorname{codim}_{T_{P_0} Y_\delta}(T_{P_0} Y_\delta \cap T_P Y_\delta) = (m - \delta + 1)(n - \delta + 1).$$

  Note that this quantity equals 1 if and only if $m = n = \delta$.
- Suppose that

  $$m - \delta + 1 > \delta - 1 \qquad \text{and} \qquad n - \delta + 1 > \delta - 1.$$

  In this case, the two zero blocks overlap in a matrix of size

  $$(m - 2\delta + 2) \times (n - 2\delta + 2).$$

  Consequently, the linear subspace $T_{P_0} Y_\delta \cap T_P Y_\delta \subseteq T_{P_0} Y_\delta$ has codimension

  $$\begin{aligned} &(m - \delta + 1)(n - \delta + 1) - (m - 2\delta + 2)(n - 2\delta + 2) \\ &= (\delta - 1)(m + n - 3\delta + 3). \end{aligned}$$

  The inequalities above give $m \geq 2\delta - 1$ and $n \geq 2\delta - 1$, so this codimension is at least $(\delta - 1)(\delta + 1) \geq 3$.

In both cases, the intersection $T_{P_0} Y_\delta \cap T_P Y_\delta$ has codimension at least 2 in $T_{P_0} Y_\delta$ under the assumption that Y_δ is not a hypersurface. This completes the proof. $\square$

4.2. **Finite linear sections with a tangency.** A c-dimensional linear subspace $L \subseteq \mathbb{P}^N$ containing P_0 is tangent to Y_δ at P_0 if and only if $L \cap T_{P_0} Y_\delta$ contains a line through P_0. The c-dimensional linear subspaces with this property form the variety

$$X_0 := \left\{ L \in \mathrm{Gr}(\mathbb{P}^c, \mathbb{P}^N) \;\middle|\; L \ni P_0,\ \dim(L \cap T_{P_0} Y_\delta) \geq 1 \right\}.$$

Our goal is to find an $L \in X_0$ that is tangent to Y_δ at P_0 with multiplicity two and transverse at all other intersection points. The proof proceeds in three steps:

(1) There exists $L \in X_0$ that is tangent to Y_δ at P_0 with multiplicity two.
(2) The locus of $L \in X_0$ that meet the singular locus $Y_{\delta-1} \subseteq Y_\delta$ has codimension ≥ 2.
(3) The locus of $L \in X_0$ that are tangent to Y_δ^{sm} at a second point has codimension ≥ 1.

Lemma 4.5. *The variety X_0 is irreducible with $\dim X_0 = cd - 1$.*

Proof. The lines in $T_{P_0} Y_\delta$ passing through P_0 form the variety

$$\mathcal{B} := \left\{ \ell \in \mathrm{Gr}(\mathbb{P}^1, T_{P_0} Y_\delta) \;\middle|\; P_0 \in \ell \right\} \cong \mathbb{P}^{d-1}.$$

Consider the incidence correspondence

$$\begin{tikzcd}[column sep=small] & I_0 := \{(L, \ell) \in X_0 \times \mathcal{B} \mid L \supseteq \ell\} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_0 & & \mathcal{B}. \end{tikzcd}$$

For each $\ell \in \mathcal{B}$, the fiber $\pi_2^{-1}(\ell) \subseteq I_0$ consists of all pairs (L, ℓ) such that $L \in \mathrm{Gr}(\mathbb{P}^c, \mathbb{P}^N)$ contains ℓ. This gives I_0 the structure of a bundle over $\mathcal{B}$ with fiber $\mathrm{Gr}(\mathbb{P}^{c-2}, \mathbb{P}^{N-2})$. It follows that I_0 is irreducible with

$$\dim I_0 = \dim \mathrm{Gr}(\mathbb{P}^{c-2}, \mathbb{P}^{N-2}) + \dim \mathcal{B} = (c-1)d + (d-1) = cd - 1.$$

The projection π_1 is surjective, so X_0 is irreducible. Fix a line $\ell \in \mathcal{B}$. A general c-plane L containing ℓ satisfies $L \cap T_{P_0} Y_\delta = \ell$. The fiber of π_1 over such an L consists of the single point (L, ℓ). Thus π_1 has zero-dimensional general fibers, and $\dim X_0 = \dim I_0 = cd - 1$. $\square$

Lemma 4.6. *There exists $L_0 \in X_0$ that is tangent to Y_δ at P_0 with multiplicity two.*

Proof. We work in the affine chart of $\mathbb{P}^N$ consisting of $m \times n$ matrices whose $(1,1)$-entry is equal to 1. Let $L_0 \subseteq \mathbb{P}^N$ be the projective linear subspace given in this chart by

$$M := \begin{pmatrix} I_{\delta-1} & A \\ B & C \end{pmatrix}$$

where

- $A = (a_{ij})$ has $a_{ij} = 0$ for all $(i,j) \neq (1,1)$,
- $B = (b_{ij})$ has $b_{11} = a_{11}$ and $b_{ij} = 0$ for all $(i,j) \neq (1,1)$,
- $C = (c_{ij})$ has $c_{11} = 0$.

The c free coordinates give $\dim L_0 = c$; setting $C = 0$ gives a line through P_0 in $T_{P_0} Y_\delta$, so $L_0 \in X_0$. By the standard block row-reduction,

$$\mathrm{rank}(M) = \mathrm{rank} \begin{pmatrix} I_{\delta-1} & A \\ 0 & C - BA \end{pmatrix} = (\delta - 1) + \mathrm{rank}(C - BA).$$

The same row reduction shows that the $\delta \times \delta$ minors generate the ideal of the entries of $C - BA$. Thus $L_0 \cap Y_\delta$ is defined in this chart by

$$b_{11}a_{11} = a_{11}^2 = 0 \qquad \text{and} \qquad c_{ij} = 0 \quad \text{for all} \quad (i, j).$$

Thus, P_0 is isolated in $L_0 \cap Y_\delta$ and has intersection multiplicity two, as required. □

Lemma 4.7. *There exists a nonempty Zariski open subset $U \subseteq X_0$ such that every $L \in U$ is tangent to Y_δ at P_0 with multiplicity two.*

Proof. Choose L_0 as in Lemma 4.6. Fix an affine chart $\mathbb{A}^N \subseteq \mathbb{P}^N$ containing P_0, with coordinates $z = (z_1, \ldots, z_N)$. Working in an affine chart of the Grassmannian containing L_0, we may choose a Zariski open neighborhood $W \subseteq X_0$ of L_0 and defining equations for the subspaces $L \in W$ of the form

$$f_{L,i}(z) = c_{i0}(L) + \sum_{j=1}^{N} c_{ij}(L) z_j = 0, \qquad \text{for } i = 1, \ldots, d,$$

where each coefficient c_{ij} is a regular function on W.

After re-indexing, we may assume that $f_{L_0,1}, \ldots, f_{L_0,d-1}$ have linearly independent differentials on $T_{P_0}Y_\delta$. This is possible since $L_0 \cap Y_\delta$ has length two at P_0, so its tangent space $L_0 \cap T_{P_0}Y_\delta$ is one-dimensional, and the d differentials $df_{L_0,i}|_{T_{P_0}Y_\delta}$ span a space of dimension exactly $d-1$. Complete $f_{L_0,1}, \ldots, f_{L_0,d-1}$ to a regular system of parameters on Y_δ at P_0 by a function t. After shrinking W, for every $L \in W$ the common zero locus of $f_{L,1}, \ldots, f_{L,d-1}$ on Y_δ in the chosen affine chart is smooth of dimension one at P_0. We may therefore choose a sufficiently small open neighborhood C_L of P_0 such that C_L is a smooth curve and $t_L := t|_{C_L}$ is a local parameter at P_0.

Since $P_0 \in L$, the restriction $f_{L,d}|_{C_L}$ vanishes at P_0, so its expansion in the local parameter t_L has no constant term. To see that the linear term also vanishes, observe that the first $d-1$ differentials $df_{L,1}(P_0), \ldots, df_{L,d-1}(P_0)$ have a one-dimensional common kernel on $T_{P_0}Y_\delta$, namely the tangent space to C_L at P_0. Since L is tangent to Y_δ at P_0, the common kernel of all d differentials on $T_{P_0}Y_\delta$ is nonzero. This kernel is contained in the one-dimensional tangent space to C_L at P_0, so the two spaces coincide. In particular, $df_{L,d}(P_0)$ vanishes on the tangent space to C_L, which means that the coefficient of t_L is zero. Consequently, for some scalar $a(L)$, we can express

$$f_{L,d}|_{C_L} \equiv a(L) t_L^2 \pmod{t_L^3}. \tag{4.2}$$

We next prove that $a(L)$ is regular. Set $x_i := f_{L_0,i}|_{Y_\delta}$ for $i = 1, \ldots, d-1$ and write $x = (x_1, \ldots, x_{d-1})^T$. The functions $x_1, \ldots, x_{d-1}, t$ are regular near P_0 on Y_δ and are fixed as L varies in W. Write $F_L = (f_{L,1}, \ldots, f_{L,d-1})^T$ and expand its restriction to Y_δ through degree two in these fixed local parameters:

$$F_L|_{Y_\delta}(x, t) = J(L)x + b(L)t + Q_L(x, t) + O(3),$$

where each entry of Q_L is homogeneous quadratic and $O(3)$ denotes terms of total degree at least three in x, t. All coefficients are regular in L, and $J(L_0)$ is the identity matrix, so we may assume that $\det J(L) \neq 0$.

Since each x_i vanishes at P_0 and t_L is a local parameter on C_L, we may write

$$x_i|_{C_L} \equiv u_i(L) t_L + v_i(L) t_L^2 \pmod{t_L^3}, \qquad \text{for } i = 1, \ldots, d-1.$$

Put

$$u(L) = (u_1(L), \dots, u_{d-1}(L))^T, \qquad v(L) = (v_1(L), \dots, v_{d-1}(L))^T.$$

To compute the quadratic contribution, use the homogeneity of Q_L:

$$Q_L(u(L)t_L, t_L) = Q_L(t_L u(L), t_L \cdot 1) = t_L^2 Q_L(u(L), 1).$$

Terms involving $v(L)t_L^2$ inside Q_L have degree at least three in t_L. Substituting the expansions into $F_L|_{C_L} = 0$ and comparing coefficients of t_L and t_L^2 gives

$$J(L)u(L) = -b(L), \qquad J(L)v(L) = -Q_L(u(L), 1).$$

After possibly shrinking W, the entries of the inverse matrix $J(L)^{-1}$ are regular on a neighborhood of L_0 in W. Then the displayed equations above show that $u(L)$ and $v(L)$ are regular in L.

Similarly, expand $f_{L,d}|_{Y_\delta}$ in these parameters:

$$f_{L,d}|_{Y_\delta}(x,t) = \sum_{i=1}^{d-1} \lambda_i(L)x_i + \beta(L)t + R_L(x,t) + O(3),$$

where R_L is homogeneous quadratic and all coefficients are regular in L. Substituting the expansions of $x_i|_{C_L}$ into (4.2) and comparing the coefficients of t_L^2, we get

$$a(L) = \sum_{i=1}^{d-1} \lambda_i(L)v_i(L) + R_L(u(L), 1),$$

which is regular in L since $u(L)$, $v(L)$, and the coefficients of this expansion are regular.

Since $a(L_0) \neq 0$, the nonvanishing locus of a is a nonempty open subset $U \subseteq X_0$. For every $L \in U$, the point P_0 is an isolated double point of $L \cap Y_\delta$. $\square$

Lemma 4.8. *The locus $X_s \subseteq X_0$ of linear subspaces $L \in X_0$ that meet the singular locus $Y_{\delta-1} \subseteq Y_\delta$ is closed and has dimension at most $cd - 3$.*

Proof. If $\delta = 2$, then Y_1 is empty and there is nothing to prove. Suppose $\delta \geq 3$. Consider the closed incidence correspondence

$$\begin{tikzcd}[column sep=small] & I_s := \{(L,P) \in X_0 \times Y_{\delta-1} \mid L \ni P\} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_0 & & Y_{\delta-1}. \end{tikzcd}$$

Since $Y_{\delta-1}$ is projective, π_1 is proper and its image X_s is closed. For each singular point $P \in Y_{\delta-1}$, every $L \in X_s$ containing P must also contain the line spanned by P_0 and P. If nonempty, the fiber $\pi_2^{-1}(P)$ can be identified with a subset of $\mathrm{Gr}(\mathbb{P}^{c-2}, \mathbb{P}^{N-2})$. Therefore,

$$\begin{aligned} \dim I_s &\leq \dim Y_{\delta-1} + \dim \mathrm{Gr}(\mathbb{P}^{c-2}, \mathbb{P}^{N-2}) \\ &= (N - (m-\delta+2)(n-\delta+2)) + (c-1)(N-c) \\ &= (d - (m+n-2\delta+3)) + (c-1)d \\ &= cd - (m+n-2\delta+3) \\ &\leq cd - 3. \end{aligned}$$

Since $X_s = \pi_1(I_s)$, we have $\dim X_s \leq \dim I_s \leq cd - 3$. $\square$

4.3. **Finite linear sections with extra tangency.** Our next goal is to bound the dimension of the locus $X_t \subseteq X_0$ defined by

$$X_t := \left\{ L \in \mathrm{Gr}(\mathbb{P}^c, \mathbb{P}^N) \;\middle|\; L \text{ is tangent at } P_0 \text{ and some } P \in Y_\delta^{\mathrm{sm}} \setminus \{P_0\} \right\}.$$

This locus is constructible: it is the image under $(L, P) \mapsto L$ of the incidence defined by $P \in L$ and $\dim(L \cap T_P Y_\delta) \geq 1$ in $X_0 \times (Y_\delta^{\mathrm{sm}} \setminus \{P_0\})$.

Lemma 4.9. *Suppose that Y_δ is not a hypersurface. Then the locus $X'_t \subseteq X_t$ defined by*

$$X'_t := \{L \in X_t \mid L \text{ is tangent at some } P \in (Y_\delta^{\mathrm{sm}} \cap T_{P_0} Y_\delta) \setminus \{P_0\}\}$$

has dimension at most $cd - 2$.

Proof. For each $P \in Y_\delta^{\mathrm{sm}} \setminus \{P_0\}$, the subspaces $L \in \mathrm{Gr}(\mathbb{P}^c, \mathbb{P}^N)$ containing both P_0 and P are precisely those that contain the line spanned by P_0 and P. Such subspaces are parametrized by $\mathrm{Gr}(\mathbb{P}^{c-2}, \mathbb{P}^{N-2})$. By Lemma 4.2, the intersection $Y_\delta^{\mathrm{sm}} \cap T_{P_0} Y_\delta$ decomposes into the strata T_i. Since $\overline{T_0}$ is a linear space containing P_0, the lines spanned by P_0 and points of $T_0 \setminus \{P_0\}$ form a family of dimension $\dim T_0 - 1 \leq d - 2$. For each nonempty T_i with $i > 0$, we have $\dim T_i \leq d - 2$, so the lines spanned by P_0 and points of T_i also form a family of dimension at most $d - 2$. Consequently,

$$\begin{aligned} \dim X'_t &\leq \dim \mathrm{Gr}(\mathbb{P}^{c-2}, \mathbb{P}^{N-2}) + (d-2) \\ &= (c-1)d + d - 2 \\ &= cd - 2. \end{aligned}$$

This yields the desired bound. $\square$

It remains to analyze the complement

$$X_t^\circ := X_t \setminus X'_t.$$

For each $L \in X_t^\circ$ and each second tangency point $P \in Y_\delta^{\mathrm{sm}} \setminus \{P_0\}$, there exist lines $\ell_0 \subseteq T_{P_0} Y_\delta$ and $\ell \subseteq T_P Y_\delta$ such that

$$P_0 \in \ell_0 \subseteq L \cap T_{P_0} Y_\delta \qquad \text{and} \qquad P \in \ell \subseteq L \cap T_P Y_\delta. \tag{4.3}$$

In view of this, we define

$$\begin{aligned} \mathcal{B}_0 &:= \left\{ \ell_0 \in \mathrm{Gr}(\mathbb{P}^1, T_{P_0} Y_\delta) \;\middle|\; P_0 \in \ell_0 \right\} \cong \mathbb{P}^{d-1}, \\ \mathcal{B}_1 &:= \left\{ (P, \ell) \in (Y_\delta^{\mathrm{sm}} \setminus \{P_0\}) \times \mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^N) \;\middle|\; P \in \ell \subseteq T_P Y_\delta \right\}. \end{aligned}$$

The projection $\mathcal{B}_1 \to Y_\delta^{\mathrm{sm}} \setminus \{P_0\}$ is a projective bundle with fiber $\mathbb{P}^{d-1}$, so $\dim \mathcal{B}_1 = 2d - 1$. Consider the incidence correspondence

$$I_t \subseteq X_t^\circ \times \mathcal{B}_0 \times \mathcal{B}_1,$$

consisting of quadruples $(L, \ell_0, (P, \ell))$ satisfying (4.3). Each such quadruple satisfies:

- The lines ℓ_0 and ℓ are distinct.
- If ℓ_0 and ℓ intersect, the intersection point is different from P.

Indeed, if either of these conditions fails, then $P \in \ell_0 \subseteq T_{P_0} Y_\delta$, contradicting the assumption that $L \in X_t^\circ$. Hence I_t decomposes as a disjoint union

$$I_t = I'_t \cup I''_t,$$

where

$$\begin{aligned} I_t' &= \{(L, \ell_0, (P, \ell)) \in I_t \mid \ell_0 \cap \ell = \emptyset\}, \\ I_t'' &= \{(L, \ell_0, (P, \ell)) \in I_t \mid \ell_0 \cap \ell = \{Q\} \text{ for some } Q \neq P\}. \end{aligned}$$

Lemma 4.10. $\dim I_t' \leq cd - 2$.

Proof. If $c < 3$, then $I_t' = \emptyset$. Suppose $c \geq 3$. Consider the projection map

$$\pi' \colon I_t' \longrightarrow \mathcal{B}_0 \times \mathcal{B}_1 : (L, \ell_0, (P, \ell)) \longmapsto (\ell_0, (P, \ell)).$$

For each quadruple $(L, \ell_0, (P, \ell)) \in I_t'$, the condition $\ell_0 \cap \ell = \emptyset$ implies that their span

$$\Pi := \langle \ell_0, \ell \rangle \subseteq \mathbb{P}^N$$

is a 3-plane. Any L occurring in such a quadruple must contain Π. Hence, over a point

$$(\ell_0, (P, \ell)) \in \mathcal{B}_0 \times \mathcal{B}_1,$$

the fiber $(\pi')^{-1}(\ell_0, (P, \ell))$ is contained in the Grassmannian of c-planes containing Π, which has dimension $(c-3)(N-c)$. It follows that

$$\begin{aligned} \dim I_t' &\leq \dim \mathcal{B}_0 + \dim \mathcal{B}_1 + (c-3)(N-c) \\ &= (d-1) + (2d-1) + (c-3)(N-c) \\ &= 3d - 2 + (c-3)d \\ &= cd - 2. \end{aligned}$$

This proves the desired upper bound. ∎

Lemma 4.11. $\dim I_t'' \leq cd - 2$, *under the assumption that Y_δ is not a hypersurface.*

Proof. Assigning to each $(L, \ell_0, (P, \ell)) \in I_t''$ the intersection point $\ell_0 \cap \ell$ defines a morphism

$$\iota \colon I_t'' \longrightarrow \mathbb{P}^N : (L, \ell_0, (P, \ell)) \longmapsto \ell_0 \cap \ell.$$

Let J_0 denote the fiber over P_0 and J_1 its complement. That is,

$$J_0 := \iota^{-1}(P_0), \qquad J_1 := I_t'' \setminus J_0.$$

We first bound the dimension of J_0. Fix a point $P \in Y_\delta^{\mathrm{sm}}$ in the image of the projection

$$\pi \colon J_0 \longrightarrow Y_\delta^{\mathrm{sm}} : (L, \ell_0, (P, \ell)) \longmapsto P.$$

The defining condition of I_t'' implies $P_0 = \ell_0 \cap \ell \neq P$. Thus, for any $(L, \ell_0, (P, \ell)) \in \pi^{-1}(P)$,

- ℓ is uniquely determined as the line through P_0 and P,
- ℓ_0 is a line in $T_{P_0} Y_\delta$ passing through P_0, and
- L is a c-dimensional linear subspace containing the plane spanned by ℓ_0 and ℓ.

The c-planes containing a fixed plane form a family of dimension $(c-2)d$; for $c = 2$, the only member is the plane itself. Thus, the above conditions imply that

$$\dim \pi^{-1}(P) \leq (c-2)d + (d-1).$$

Moreover, we have $P_0 \in \ell \subseteq T_P Y_\delta$, so in particular $T_P Y_\delta$ contains P_0. By Lemma 4.3, the locus of such points $P \in Y_\delta^{\mathrm{sm}}$ has dimension at most $d-1$. Combining this with the bound for each nonempty fiber, we obtain

$$\dim J_0 \leq \big((c-2)d + (d-1)\big) + (d-1) = cd - 2.$$

Next, we bound the dimension of J_1. Consider the projection

$$\rho\colon J_1 \longrightarrow Y_\delta^{\mathrm{sm}} : (L, \ell_0, (P, \ell)) \longmapsto P.$$

For each $P \in Y_\delta^{\mathrm{sm}}$, the restriction of ι to the fiber $\rho^{-1}(P) \subseteq J_1$ defines a map

$$\iota_P := \iota|_{\rho^{-1}(P)} \colon \rho^{-1}(P) \longrightarrow T_{P_0}Y_\delta \cap T_P Y_\delta \cong \mathbb{P}^e.$$

By the defining conditions of I_t'' and J_1, we have

$$P_0 \neq \ell_0 \cap \ell \neq P.$$

Thus, for each $Q \in \operatorname{im}(\iota_P)$, the fiber $\iota_P^{-1}(Q)$ consists of $(L, \ell_0, (P, \ell)) \in \rho^{-1}(P)$ such that

- ℓ_0 is uniquely determined as the line through P_0 and Q,
- ℓ is uniquely determined as the line through P and Q, and
- L is a c-dimensional linear subspace containing the plane spanned by ℓ_0 and ℓ.

These imply that each nonempty fiber satisfies

$$\dim \rho^{-1}(P) \leq (c-2)d + e.$$

For $P \in \operatorname{im}(\rho)$, the definition of X_t° gives $P \notin T_{P_0}Y_\delta$. Since $P \in T_P Y_\delta$, the two tangent spaces are distinct, so $e \leq d-1$.

Consider the locus

$$Z := \left\{ P \in Y_\delta^{\mathrm{sm}} \mid \dim\left(T_{P_0}Y_\delta \cap T_P Y_\delta\right) \geq d-1 \right\}.$$

Since Y_δ is not a hypersurface, Lemma 4.4 shows that Z is a proper subset of Y_δ^{sm}. It is closed by upper semicontinuity, so $\dim Z \leq d-1$. Consequently,

$$\begin{aligned}
\dim \rho^{-1}(Y_\delta^{\mathrm{sm}} \setminus Z) &\leq d + (c-2)d + (d-2) = cd - 2, \\
\dim \rho^{-1}(Z) &\leq (d-1) + (c-2)d + (d-1) = cd - 2.
\end{aligned}$$

Therefore $\dim J_1 \leq cd - 2$. It follows that

$$\dim I_t'' \leq \max\{\dim J_0,\, \dim J_1\} \leq cd - 2,$$

which proves the desired bound. ☐

Lemma 4.12. $\dim X_t \leq cd - 2$, *under the assumption that Y_δ is not a hypersurface.*

Proof. Due to Lemma 4.9, it remains to prove that

$$\dim X_t^\circ \leq cd - 2.$$

Each $L \in X_t^\circ$ is tangent at both P_0 and some point $P \in Y_\delta^{\mathrm{sm}} \setminus \{P_0\}$. Hence there exist lines

$$\ell_0 \subseteq L \cap T_{P_0}Y_\delta \qquad \text{and} \qquad \ell \subseteq L \cap T_P Y_\delta$$

passing through P_0 and P, respectively. This shows that the projection map

$$I_t \longrightarrow X_t^\circ : (L, \ell_0, (P, \ell)) \longmapsto L$$

is surjective. It follows that

$$\dim X_t^\circ \leq \dim I_t \leq \max\{\dim I_t',\, \dim I_t''\},$$

which is at most $cd - 2$ by Lemmas 4.10 and 4.11. ☐

Theorem 4.13. *Every determinantal variety $Y_\delta \subseteq \mathbb{P}^N$ is quasireflexive. That is, there exists a linear subspace $L \subseteq \mathbb{P}^N$ of dimension $N - \dim Y_\delta$ that intersects Y_δ in $\deg(Y_\delta) - 1$ points at which Y_δ is smooth.*

Proof. The hypersurface case is proved in Proposition 4.1. Assume that Y_δ is not a hypersurface. By Lemma 4.7, there exists a Zariski open subset $U \subseteq X_0$ such that $Y_\delta \cap L$ has multiplicity exactly two at P_0 for each $L \in U$.

The constructible locus X_t and its closure have the same dimension. By Lemmas 4.5, 4.8 and 4.12, $X_s \cup \overline{X_t}$ is a proper closed subset of the irreducible variety X_0. Choose any

$$L \in U \setminus (X_s \cup \overline{X_t}).$$

Then the intersection $L \cap Y_\delta$ is disjoint from the singular locus $Y_{\delta-1}$ and is transverse away from the double point P_0. Since all intersection points are isolated, $L \cap Y_\delta$ is finite. By Bézout's theorem, the total intersection multiplicity is $\deg(Y_\delta)$. The point P_0 contributes two, whereas every other intersection point contributes one. Hence $L \cap Y_\delta$ is supported on exactly $\deg(Y_\delta) - 1$ distinct smooth points, as desired. $\square$

Appendix A. Reflexivity of determinantal varieties

For irreducible projective curves of degree at least two, reflexivity and quasireflexivity are equivalent in characteristic different from 2 [Ent21, Proposition 2.1]. In higher dimensions, following the second paragraph of the proof of [BH86, Theorem at p. 906], one deduces that:

Proposition A.1. *Let K be an algebraically closed field of characteristic different from 2, and let $Y \subseteq \mathbb{P}^N_K$ be an irreducible projective variety of degree at least two. If Y is reflexive, then Y is quasireflexive.*

In what follows, we prove that determinantal varieties defined over a field of arbitrary characteristic are reflexive. By Proposition A.1, this already implies their quasireflexivity in characteristic different from 2. However, it does not address the case of characteristic 2, which is of particular importance in coding theory.

We begin by recalling the definition of reflexivity. Let $Y \subseteq \mathbb{P}^N$ be a projective variety defined over an algebraically closed field K of arbitrary characteristic. Its *conormal variety* $C(Y) \subseteq \mathbb{P}^N \times \left(\mathbb{P}^N\right)^*$ is the Zariski closure of the set of pairs $(P, H) \in \mathbb{P}^N \times \left(\mathbb{P}^N\right)^*$ where

- P belongs to the smooth locus $Y^{\mathrm{sm}} \subseteq Y$, and
- H is a hyperplane in $\mathbb{P}^N$ containing the tangent space $T_P Y \subseteq \mathbb{P}^N$.

Consider the two projection maps

$$\begin{tikzcd} & C(Y) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ Y & & \left(\mathbb{P}^N\right)^* . \end{tikzcd}$$

The *dual variety* of Y is the image $Y^* := \pi_2(C(Y)) \subseteq (\mathbb{P}^N)^*$ of the second projection, and Y is *reflexive* if $C(Y) \cong C(Y^*)$ under the natural isomorphisms

$$\mathbb{P}^N \times \left(\mathbb{P}^N\right)^* \cong \left(\mathbb{P}^N\right)^* \times \mathbb{P}^N \cong \left(\mathbb{P}^N\right)^* \times \left(\mathbb{P}^N\right)^{**}. \tag{A.1}$$

Let $Y_\delta \subseteq \mathbb{P}^N = \mathbb{P}^{mn-1}$ be the determinantal variety over K consisting of $m \times n$ matrices of rank $< \delta$, where $2 \leq \delta \leq n \leq m$. Recall that the tangent space at a smooth point $P \in Y_\delta^{\mathrm{sm}}$ is given by

$$T_P Y_\delta = \left\{ Q \in \mathbb{P}^N \;\middle|\; Q(\ker P) \subseteq \operatorname{im} P \right\}.$$

On the other hand, there is a nondegenerate bilinear form

$$\langle -, - \rangle \colon K^{mn} \times K^{mn} \longrightarrow K, \quad (A, B) \longmapsto \operatorname{tr}(AB^T).$$

In coordinates, if we write $A = (a_{ij})$ and $B = (b_{ij})$, then $\langle A, B\rangle = \sum_{i,j} a_{ij} b_{ij}$. We identify $(\mathbb{P}^N)^*$ with $\mathbb{P}^N$ via the induced isomorphism, which is explicitly given by

$$\mathbb{P}^N \xrightarrow{\sim} (\mathbb{P}^N)^* : P \longmapsto H_P = \left\{ \sum_{i,j} p_{ij} x_{ij} = 0 \right\}$$

where (p_{ij}) is any matrix representing P.

Lemma A.2. *Let $I_{\delta-1}$ denote the $(\delta-1)\times(\delta-1)$ identity matrix. The fiber over*

$$P_0 = \begin{pmatrix} I_{\delta-1} & 0 \\ 0 & 0 \end{pmatrix} \in Y_\delta$$

under the projection $\pi_1 \colon C(Y_\delta) \longrightarrow Y_\delta$ has the form

$$\pi_1^{-1}(P_0) = \left\{ \left(P_0, \begin{pmatrix} 0 & 0 \\ 0 & *_{m-\delta+1,\, n-\delta+1} \end{pmatrix} \right) \right\}$$

where $*_{\alpha,\beta}$ ranges over nonzero $\alpha\times\beta$ matrices, considered up to scalar. In particular,

$$\pi_2\left(\pi_1^{-1}(P_0)\right) \subseteq Y_{n-\delta+2}.$$

Proof. Every $Q \in T_{P_0} Y_\delta$ is represented by a matrix of the form

$$\begin{pmatrix} *_{\delta-1,\,\delta-1} & *_{\delta-1,\,n-\delta+1} \\ *_{m-\delta+1,\,\delta-1} & 0 \end{pmatrix}.$$

Hence, H_P contains $T_{P_0} Y_\delta$ if and only if P is represented by

$$\widetilde{P} = \begin{pmatrix} 0 & 0 \\ 0 & *_{m-\delta+1,\,n-\delta+1} \end{pmatrix}.$$

Since $(P_0, P) \in \mathbb{P}^N \times (\mathbb{P}^N)^*$ belongs to $\pi_1^{-1}(P_0)$ if and only if $T_{P_0} Y_\delta \subseteq H_P$, this proves the first statement. The second statement follows immediately from the expression of $\widetilde{P}$. $\square$

Lemma A.3. *For any $P \in Y_\delta^{\mathrm{sm}}$, $Q \in \mathbb{P}^N$, $A \in \mathrm{GL}_m(K)$ and $B \in \mathrm{GL}_n(K)$,*

$$T_P Y_\delta \subseteq H_Q \iff T_{APB} Y_\delta \subseteq H_{A^\dagger Q B^\dagger}$$

where $A^\dagger = (A^{-1})^T$ and $B^\dagger = (B^{-1})^T$.

Proof. Note that

$$\mathrm{tr}\left(P\, Q^T\right) = \mathrm{tr}\left((APB)\,(A^\dagger Q B^\dagger)^T\right). \tag{A.2}$$

Using the identities

- $\ker(APB) = \ker(PB) = B^{-1} \ker P$,
- $\mathrm{im}(APB) = \mathrm{im}(AP) = A\ \mathrm{im}\, P$,

we deduce for any $M \in \mathrm{Hom}(K^n, K^m)$ that

$$\begin{aligned} (AMB)\ \ker(APB) \subseteq \mathrm{im}(APB) &\iff (AMB)\,(B^{-1}\ker P) \subseteq A\ \mathrm{im}\, P \\ &\iff M\ \ker P \subseteq \mathrm{im}\, P. \end{aligned}$$

It follows that

$$A\,(T_P Y_\delta)\,B = T_{APB} Y_\delta. \tag{A.3}$$

Consequently,

$$T_P Y_\delta \subseteq H_Q \iff \langle T_P Y_\delta,\, Q\rangle = 0$$

$$\text{(by (A.2))} \qquad \iff \left\langle A\,(T_PY_\delta)\,B,\ A^\dagger Q B^\dagger\right\rangle = 0$$
$$\text{(by (A.3))} \qquad \iff \left\langle T_{APB}Y_\delta,\ A^\dagger Q B^\dagger\right\rangle = 0 \iff T_{APB}Y_\delta \subseteq H_{A^\dagger Q B^\dagger}.$$

This completes the proof. □

Lemma A.4. *Let $P \in Y_\delta^{\mathrm{sm}}$ and $Q \in \mathbb{P}^N$. Then*

$$T_PY_\delta \subseteq H_Q \quad\iff\quad PQ^T = 0 \quad \textit{and} \quad P^TQ = 0.$$

Proof. Choose $A \in \mathrm{GL}_m(K)$ and $B \in \mathrm{GL}_n(K)$ such that $APB = P_0$, and set

$$Q' := A^\dagger Q B^\dagger, \qquad A^\dagger = (A^{-1})^T, \qquad B^\dagger = (B^{-1})^T.$$

By Lemma A.3,

$$T_PY_\delta \subseteq H_Q \quad\iff\quad T_{P_0}Y_\delta \subseteq H_{Q'}.$$

By Lemma A.2, the latter condition holds if and only if Q' has only a lower-right block. This is equivalent to

$$P_0(Q')^T = 0 \qquad \text{and} \qquad P_0^TQ' = 0.$$

On the other hand,

$$P_0(Q')^T = (APB)(A^\dagger QB^\dagger)^T = A(PQ^T)A^{-1}$$

and

$$P_0^TQ' = (APB)^T(A^\dagger QB^\dagger) = B^T(P^TQ)(B^T)^{-1}.$$

Thus, the last condition is equivalent to $PQ^T = 0$ and $P^TQ = 0$. □

Proposition A.5. *For any $(P,Q) \in Y_\delta^{\mathrm{sm}} \times Y_{n-\delta+2}^{\mathrm{sm}}$,*

$$T_PY_\delta \subseteq H_Q \quad\iff\quad T_QY_{n-\delta+2} \subseteq H_P.$$

In particular, Y_δ has dual variety $Y_{n-\delta+2}$ and is reflexive.

Proof. Set $\delta' = n - \delta + 2$. By Lemma A.4,

$$T_PY_\delta \subseteq H_Q \quad\iff\quad PQ^T = 0 \quad \text{and} \quad P^TQ = 0.$$

Applying the same lemma with P and Q reversed gives

$$T_QY_{\delta'} \subseteq H_P \quad\iff\quad QP^T = 0 \quad \text{and} \quad Q^TP = 0.$$

Therefore, the condition $T_PY_\delta \subseteq H_Q$ is equivalent to $T_QY_{\delta'} \subseteq H_P$.

Interchanging the factors gives an isomorphism

$$\{(P,Q) \in Y_\delta^{\mathrm{sm}} \times Y_{\delta'}^{\mathrm{sm}} \mid T_PY_\delta \subseteq H_Q\} \cong \{(Q,P) \in Y_{\delta'}^{\mathrm{sm}} \times Y_\delta^{\mathrm{sm}} \mid T_QY_{\delta'} \subseteq H_P\}.$$

By Lemmas A.2 and A.3, each fiber over Y_δ^{sm} under the map $C(Y_\delta) \to Y_\delta$ is the projective space of $(m-\delta+1) \times (n-\delta+1)$ matrices. Since $m \geq n$, matrices of full column rank form a nonempty open dense subset. Within each fiber, these correspond precisely to the points with $Q \in Y_{\delta'}^{\mathrm{sm}}$. Thus, the incidence locus on the left is dense in $C(Y_\delta)$. As $n - \delta' + 2 = \delta$, the same argument shows that the locus on the right is dense in $C(Y_{\delta'})$.

Taking closures shows that interchanging the factors identifies $C(Y_\delta)$ with $C(Y_{\delta'})$. Projecting to the factors gives $Y_\delta^* = Y_{\delta'}$ and $Y_{\delta'}^* = Y_\delta$. In particular, $C(Y_\delta) \cong C(Y_\delta^*)$, so Y_δ is reflexive. □

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